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   This paper shows that the stability of these modes is general feature of the dispersion relation, and can be expected to be stable for all anisotropic velocity distributions.

2. Two-Dimensional Nonlinear Schrodinger Equation and Self-Focusing in a Two-Fluid Model of Newtonian Cosmological Perturbations, by Ronald E. Kates and D. J. Kaup [Astron. Astrophys. 206, 9-17 (1988)].
   This manuscript describes how nonlinear evolution could be an important contribution to the cosmological expansion.

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   This paper presents the coefficients for a nonlinear theory of a planar magnetron. We are currently numerically evaluating these coefficients to see how they correspond to the operating range of an actual device.

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   This paper give our final result for the forced integrable systems. We can show that given one function, the entire problem becomes solvable. The unknown is how to generate this function from the boundary values.

In addition to the above, there are also the following preprints in various stages of being accepted for publication.

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   This paper describes how to generate the fundamental singular solutions of an integrable nonlinear elliptic PDE.

3. Coherent Structures in the Planar Magnetron, by D.J. Kaup [submitted to Phys. of Fluids]
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   This paper demonstrates a potential danger in large scale particle simulations. Due to the necessity in numerical simulations to approximate a large number of particles by one particle, it is then possible to excite thermal instabilities. This is simply because the numberical statistics will always be worst than in the actual situation, since the number of particles are less. Thus the numberical calculations can have a much higher "temperature", which could then cause a thermal instability to be excited in the numerical calculations. And this thermal instability may never be excited in the actual situation because of the much lower actual temperature.

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   Curtis Menyuk devised a very simple model to test my statements that soliton eigenvalues could move in SRS. The model is a remarkable eye-opener. Not only do these soliton's eigenvalues move, but they grow without limit! Thus one can mathematically treat SRS as a pure N-soliton solution, but with N = infinity.

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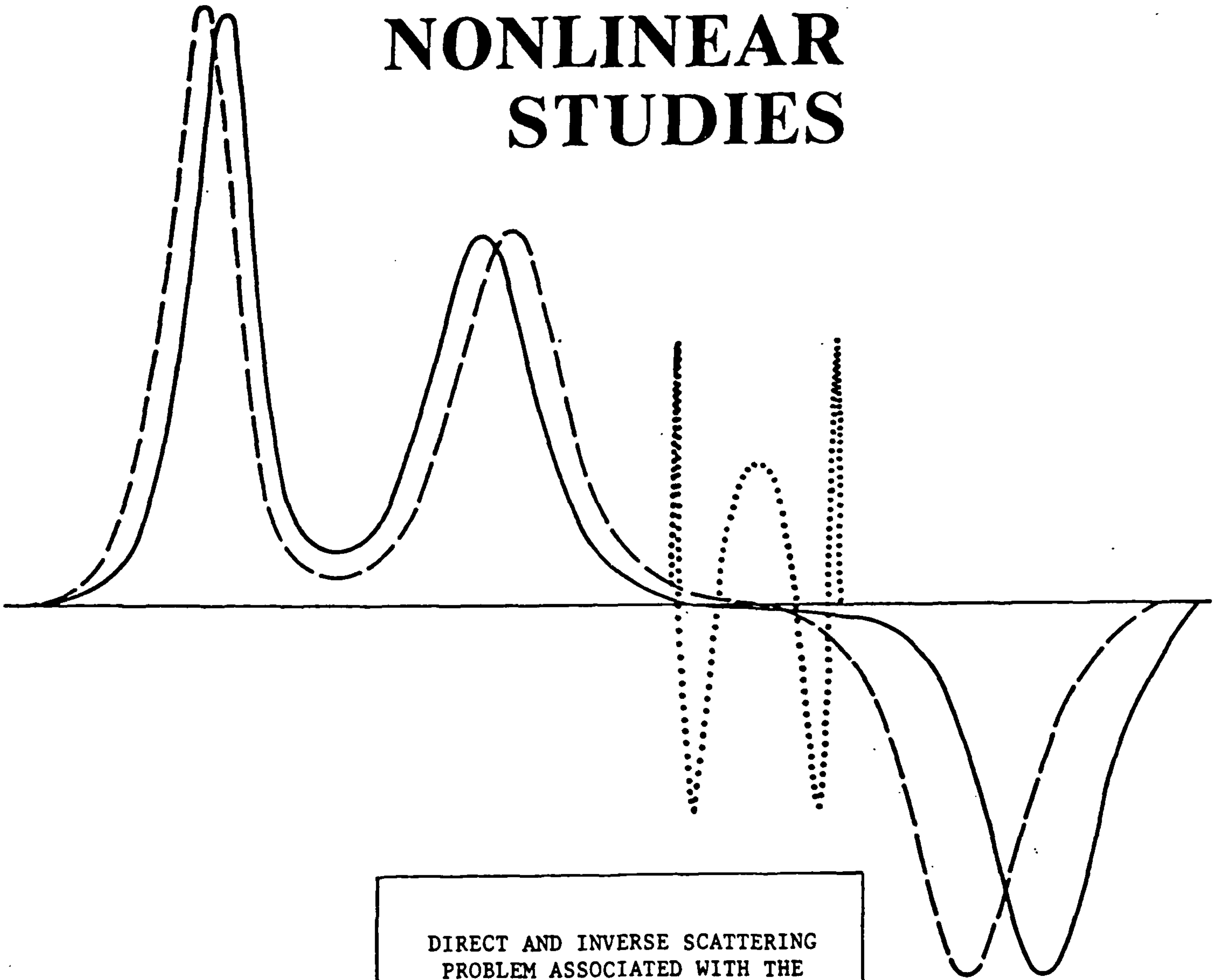

DIRECT AND INVERSE SCATTERING PROBLEM ASSOCIATED WITH THE ELLIPTIC SINH-GORDON EQUATION

BY

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September 26, 1989

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Abstract

The direct and inverse scattering problem is solved for the elliptic Sinh-Gordon equation. It is shown that the inverse scattering transform may be useful in the analysis of localized singular solutions. As an example, a cylindrically symmetric singular solution is discussed in detail.

1 Introduction

The elliptic Sinh-Gordon equation

$$u_{xx} + u_{yy} = \lambda^2 \sinh u \tag{1}$$

appears in plasma physics [1-5] as a two dimensional model equation describing a system of interacting charged particles. Depending on the sign of λ^2 we can consider both positive and negative temperature states of thermal equilibrium [4], corresponding to essentially different solutions of eq. (1).

Negative temperature states ($\lambda^2 < 0$) have been studied extensively [3-5], and both numerical and analytical solutions have been reported, exhibiting stable nonuniform distribution of the potential and charge density within a bounded region.

On the other hand, little attention has been paid so far to the positive temperature states ($\lambda^2 > 0$), perhaps because of the fact that the only possible regular solution (if it exists) is trivial, i.e. corresponds to the uniform distribution of charge density.

The case $\lambda^2 > 0$ becomes nontrivial if we place a fixed charge (or a number of charges) into the medium described by eq. (1), i.e. if we allow the potential to be singular at some point(s) within the region of interest.

In this paper we confine our attention to this latter case, and without loss of generality we put $\lambda^2 = 1$, by rescaling eq. (1) to the dimensionless form

$$u_{xx} + u_{yy} = \sinh u \ . \tag{2}$$

A linearized version of eq. (2)

$$u_{xx} + u_{yy} - u = 0 \tag{3}$$

can be easily solved using the Green function formalism [6]. Indeed, for a unit point source at $\vec{r}$ and the boundary conditions $u \to 0$ as $|\vec{r}| \to \infty$, the Green function is given by

$$G(\vec{r}, \vec{r}\,') = K_0(|\,\vec{r} - \vec{r}\,'\,|) \quad , \tag{4}$$

where $K_0(r)$ denotes the modified Bessel function of the second kind and order 0, $\vec{r}, \vec{r}\,'$ are vectors with the components $(x, y), (x', y',)$, respectively.

Placing a point charge of strength A at the origin we find a cylindrically symmetric solution

$$u(r) = AK_0(r), \quad r = |\,\vec{r}\,| = \sqrt{x^2 + y^2} \quad , \tag{5}$$

which satisfies eq. (3) everywhere, except for the singularity.

On the other hand, in the strong nonlinearity limit we can expect $u \to \infty$ as $r \to 0$, and the solution of (2) tending to that of the Liouville equation

$$u_{xx} + u_{yy} = \frac{1}{2} e^u \quad . \tag{6}$$

The general solution of eq. (6) is given in terms of two arbitrary functions [7]. Imposing cylindrical symmetry we can find a class of solutions singular at the origin, with the leading terms given by

$$u = -\alpha \ln r + \beta + O(r) \quad , \tag{7}$$

where $\alpha \in (0, 2)$.

Unfortunately, in contrast to eqs. (3) and (6), a closed-form solution of eq. (2) having cylindrical symmetry is not known. Naturally, one can use numerical methods to integrate (2), starting e.g. from the asymptotic expression (5). In this paper, however, we apply the inverse scattering transform (IST) [8-11] in order to reconstruct the solution of (2) for arbitrary $\vec{r}$.

It should be noted here, that the elliptic problems are, in general, well-posed when the boundary conditions are imposed along a closed curve surrounding the region of interest [6]. On the other hand, the initial value problem, which is a natural choice for the hyperbolic equation, may be unstable (ill-conditioned) when applied to the elliptic case. In this context, we could expect possible difficulties in the application of the IST to an elliptic problem. In fact, the IST has always before been used only for solving evolution equations of the hyperbolic type, and to the authors knowledge it has not been adapted to the elliptic case.

Therefore, the main aim of this paper is to study the applicability and usefulness of the IST formalism to the analysis of elliptic problems. In this connection, in Section 2 we consider the direct scattering problem in a rather general case, without specifying details of the solution. In particular, the analytic properties of the scattering data are determined by following closely the methods of Refs. [9,11]. The y-dependence of the scattering data is discussed in Section 3, while Section 4 deals with the inverse scattering problem. In Section 5 we consider a cylindrically symmetric solution of eq. (2), having singularity at the origin and vanishing exponentially as $r \to \infty$. Section 6 contains concluding remarks, and we shall also point out some open problems.

2 Analytical Properties of the Scattering Data

Following [8,10], let us consider the linear eigenvalue problem

$$v_{1,x} = -i\left(\frac{\zeta}{2} - \frac{\cosh u}{8\zeta}\right)v_1 - \left(p_+ + \frac{\sinh u}{8\zeta}\right)v_2 \ , \tag{8a}$$

$$v_{2,x} = \left(p_+ - \frac{\sinh u}{8\zeta}\right)v_1 + i\left(\frac{\zeta}{2} - \frac{\cosh u}{8\zeta}\right)v_2 \ , \tag{8b}$$

where $p_+ = \frac{1}{4}(iu_x + u_y)$ and ζ is the spectral parameter. Take the y-dependence of the eigenfunctions v_1, v_2 to be

$$v_{1,y} = \left(\frac{\zeta}{2} + \frac{\cosh u}{8\zeta}\right)v_1 - i\left(p_+ - \frac{\sinh u}{8\zeta}\right)v_2 \ , \tag{9a}$$

$$v_{2,y} = i\left(p_+ + \frac{\sinh u}{8\zeta}\right)v_1 - \left(\frac{\zeta}{2} + \frac{\cosh u}{8\zeta}\right)v_2 \ . \tag{9b}$$

Then it can be verified by cross-differentiation that (8a,b) and (9a,b) are compatible if: (i) u satisfies eq. (2) and (ii) ζ is independent of y.

For ζ real and u tending to zero sufficiently rapidly as $x \to \pm\infty$ we define the solutions of (8a,b) with the asymptotic form:

$$\phi(x,\zeta) \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-ikx} \quad \text{as} \quad x \to -\infty \ , \tag{10a}$$

$$\psi(x,\zeta) \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{ikx} \quad \text{as} \quad x \to +\infty \ , \tag{10b}$$

where $k(\zeta) = \frac{\zeta}{2} - \frac{1}{8\zeta}$.

It can be shown that if

$$v(x,\zeta) = \begin{bmatrix} v_1(x,\zeta) \\ v_2(x,\zeta) \end{bmatrix}$$

is a solution of (8a,b), then

$$\bar{v}(x,\zeta) = \begin{bmatrix} v_2(x, -\zeta) \\ -v_1(x, -\zeta) \end{bmatrix} \tag{11}$$

is also a solution.

In particular, we have

$$\bar{\phi}(x,\zeta) \to \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{ikx} \quad \text{as} \quad x \to -\infty \tag{12a}$$

and

$$\bar{\psi}(x,\zeta) \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-ikx} \quad \text{as} \quad x \to +\infty \ . \tag{12b}$$

Since ψ and $\bar{\psi}$ are linearly independent, we can write for ζ real

$$\phi(x,\zeta) = a(\zeta)\bar{\psi}(x,\zeta) + b(\zeta)\psi(x,\zeta) \ , \tag{13a}$$

$$\bar{\phi}(x,\zeta) = \bar{b}(\zeta)\bar{\psi}(x,\zeta) - \bar{a}(\zeta)\psi(x,\zeta) \ , \tag{13b}$$

where

$$a(\zeta)\bar{a}(\zeta) + b(\zeta)\bar{b}(\zeta) = 1 \ . \tag{14}$$

Consequently, it follows from (11) that

$$\bar{a}(\zeta) = a(-\zeta) \ , \tag{15a}$$

$$\bar{b}(\zeta) = b(-\zeta) \ . \tag{15b}$$

One can also show that

$$a\left(\frac{1}{4\zeta}\right) = a^*(\zeta^*) \ , \qquad (16a)$$

$$b\left(\frac{1}{4\zeta}\right) = b^*(\zeta^*) \ , \qquad (16b)$$

and if $u(x,-y) = u(x,y)$ then

$$\bar{a}(y,\ \zeta) = a^*(-y,\ \zeta^*) \ , \qquad (17a)$$

$$\bar{b}(y,\ \zeta) = -b^*(-y,\ \zeta^*) \ , \qquad (17b)$$

the last two relations being particularly useful when the solution of eq. (2) is symmetric with respect to the y-axis.

In order to determine the analytical properties of the eigenfunctions ϕ, ψ we replace the differential equation (8) satisfying the boundary conditions (10) by an equivalent integral equation [9,11]

$$\phi_1(x,\zeta)e^{i\alpha(x,\zeta)} = 1 + \int_{-\infty}^{x} N_1(x,z,\zeta)\phi_1(z,\zeta)e^{i\alpha(z,\zeta)}dz \ , \qquad (18a)$$

$$\phi_2(x,\zeta)e^{i\alpha(x,\zeta)} = \int_{-\infty}^{x} N_2(x,z,\zeta)\phi_1(z,\zeta)e^{i\alpha(z,\zeta)}dz \ , \qquad (18b)$$

where

$$N_1(x,z,\zeta) = q(z)\int_{z}^{x} r(t)e^{2i[\alpha(t,\zeta)-\alpha(z,\zeta)]}dt \ ,$$

$$N_2(x,z,\zeta) = q(z)e^{2i[\alpha(x,\zeta)-\alpha(z,\zeta)]} \ ,$$

$$q(x) = p_+(x) - \frac{\sinh u(x)}{8\zeta}, \quad r(x) = -\left(p_+(x) + \frac{\sinh u(x)}{8\zeta}\right) \ ,$$

$$\alpha(x,\zeta) = k(\zeta)x - \frac{1}{4\zeta}\int_{-\infty}^{x} \sinh^2 \frac{u(z)}{2} dz \ .$$

Extending $\phi_1(x,\zeta)$ into the upper half plane ($\zeta = \xi + i\eta, \eta > 0$) one can show that for $|\ \zeta\ | \geq \frac{1}{2}$

$$|\ \phi_1(x,\zeta)e^{i\alpha(x,\zeta)}\ | \leq e^{I(x)} \cosh V_+(x) \ , \tag{19a}$$

and

$$|\ \phi_2(x,\zeta)e^{i\alpha(x,\zeta)}\ | \leq e^{I(x)} \sinh V_+(x) \ , \tag{19b}$$

where

$$I(x) = \frac{1}{2}\int_{-\infty}^{x} \sinh^2 \frac{u(z)}{2} dz$$

$$V_+(x) = \int_{-\infty}^{x} \left\{|\ p_+(z)\ | + \frac{1}{4}\ |\ \sinh u(z)\ |\right\} dz \ .$$

For $|\ \zeta\ | \leq \frac{1}{2}$ we should consider an alternative form of the eigenvalue problem [9]. Proceeding as before, we find for $\eta > 0$

$$|\ \phi_1(x,\zeta)e^{ikx}\ | \leq e^{I(x)}\left\{|\ \cosh\frac{u}{2}\ |\ \cosh V_-(x) + |\ \sinh\frac{u}{2}\ |\ \sinh V_-(x)\right\} \ , \tag{20a}$$

$$|\ \phi_2(x,\zeta)e^{ikx}\ | \leq e^{I(x)}\left\{|\ \cosh\frac{u}{2}\ |\ \sinh V_-(x) + |\ \sinh\frac{u}{2}\ |\ \cosh V_-(x)\right\} \ , \tag{20b}$$

where $V_-(x) = \int_{-\infty}^{x} \left\{ | p_-(z) | + \frac{1}{4} | \sinh u(z) | \right\} dz$ and $p_- = \frac{1}{4}(iu_x - u_y)$.

Differentiating $\phi(x,\zeta)e^{ikx}$ with respect to ζ we can see that the derivative exists for $\eta > 0$. Thus, we can summarize the analytical properties of eigenfunctions ϕ, ψ and scattering coefficients a, b:

Theorem - If the following integrals

$$\int_{-\infty}^{+\infty} | u_x | \, dx, \int_{-\infty}^{+\infty} | u_y | \, dx, \int_{-\infty}^{+\infty} | \sinh u | \, dx$$

are bounded, then $\phi(x,\zeta)e^{ikx}, \psi(x,\zeta)e^{-ikx}$ and $a(\zeta)$ are analytic in the upper half plane ($\eta > 0$). For $\eta = 0$, the above functions, as well as $b(\zeta)$ are at least bounded.

For $\zeta \to \infty$ in the upper half plane one can find that

$$\phi(x,\zeta)e^{ikx} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + O\left(\frac{1}{\zeta}\right) \quad , \tag{21a}$$

$$\psi(x,\zeta)e^{-ikx} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + O\left(\frac{1}{\zeta}\right) \tag{21b}$$

and

$$a(\zeta) = 1 + O\left(\frac{1}{\zeta}\right) \quad . \tag{21c}$$

Similarly, for $\zeta \to 0$ in the upper half plane we have

$$\phi(x,\zeta)e^{ikx} = \begin{bmatrix} \cosh \frac{u}{2} \\ i \sinh \frac{u}{2} \end{bmatrix} + O(\zeta) \quad , \tag{22a}$$

$$\psi(x,\zeta)e^{-ikx} = \begin{bmatrix} -i \sinh \frac{u}{2} \\ \cosh \frac{u}{2} \end{bmatrix} + O(\zeta) \quad , \tag{22b}$$

$$a(\zeta) = 1 + O(\zeta) \ . \tag{22c}$$

3 The y-dependence of the scattering data

Eqs. (9a,b) can be written in a matrix form as

$$v_y = \begin{bmatrix} A & B \\ C & -A \end{bmatrix} v \ , \tag{23}$$

where

$$A(x,y,\zeta) = \frac{\zeta}{2} + \frac{\cosh u(x,y)}{8\zeta} \ ,$$

$$B(x,y,\zeta) = i\left(\frac{\sinh u(x,y)}{8\zeta} - p_+(x,y)\right) \ ,$$

$$C(x,y,\zeta) = i\left(\frac{\sinh u(x,y)}{8\zeta} + p_+(x,y)\right) \ . \tag{24}$$

Since $u(x,y) \to 0$ as $\mid x \mid \to \infty$, we have

$$\lim_{|x|\to\infty} A(x,y,\zeta) = A_0(\zeta) = \frac{\zeta}{2} + \frac{1}{8\zeta} \ ,$$

$$\lim_{|x|\to\infty} B(x,y,\zeta) = \lim_{|x|\to\infty} C(x,y,\zeta) = 0 \ . \tag{25}$$

Eq.(23) is satisfied by the functions $\phi(x,y,\zeta)e^{A_0 y}$ and $\overline{\phi}(x,y,\zeta)e^{-A_0 y}$, thus the corresponding equations for ϕ and $\overline{\phi}$ can be written as

$$\phi_y = \begin{bmatrix} A - A_0 & B \\ C & -A - A_0 \end{bmatrix} \phi \tag{26a}$$

and

$$\overline{\phi}_y = \begin{bmatrix} A + A_0 & B \\ C & -A + A_0 \end{bmatrix} \overline{\phi} \ . \tag{26b}$$

Taking the limit $x \to +\infty$ in (26a,b), we find the differential equations for the scattering coefficients:

$$a_y(y, \zeta) = 0 \ , \tag{27a}$$

$$b_y(y, \zeta) = 2A_0 b(y, \zeta) \ , \tag{27b}$$

$$\overline{a}_y(y, \zeta) = 0 \ , \tag{27c}$$

$$\overline{b}_y(y, \zeta) = -2A_0 \overline{b}(y, \zeta) \ . \tag{27d}$$

Thus

$$a(y, \zeta) = a(0, \zeta) \ , \tag{28a}$$

$$b(y, \zeta) = b(0, \zeta) e^{-2A_0(\zeta) y} \ , \tag{28b}$$

$$\overline{a}(y, \zeta) = \overline{a}(0, \zeta) \ , \tag{28c}$$

$$\overline{b}(y, \zeta) = \overline{b}(0, \zeta) e^{2A_0(\zeta) y} \ . \tag{28d}$$

4 The Inverse Scattering Problem

Following [9] we assume $\psi(x,\zeta)$ to be given by

$$\psi(x,\zeta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{ikx} + \int_x^\infty \left[\mathcal{K}(x,s) + \frac{i}{2\zeta} M(x) L(x,s)\right] e^{iks} ds \ , \qquad (29)$$

where

$$M(x) = \begin{bmatrix} \cosh\frac{u}{2} & i\sinh\frac{u}{2} \\ i\sinh\frac{u}{2} & -\cosh\frac{u}{2} \end{bmatrix} \qquad (30)$$

and $\mathcal{K}$, L are independent of ζ.

Substituting (29) into the eigenvalue problem (8a,b) and taking the Fourier transform we obtain the following differential equations for $\mathcal{K}$ and L:

$$[I\partial_x - \sigma_3\partial_s + i\sigma_2 p_+(x)]\,\mathcal{K}(x,s) + \frac{1}{2}\sigma_2 \sinh\frac{u(x)}{2} L(x,s) = 0 \ , \qquad (31a)$$

$$[I\partial_x - \sigma_3\partial_s + i\sigma_2 p_-(x)]\,L(x,s) + \frac{1}{2}\sigma_2 \sinh\frac{u(x)}{2} \mathcal{K}(x,s) = 0 \ , \qquad (31b)$$

subject to the boundary conditions:

$$\lim_{s\to\infty} \mathcal{K}(x,s) = 0 \ , \qquad (32a)$$

$$\lim_{s\to\infty} L(x,s) = 0 \ , \qquad (32b)$$

$$\mathcal{K}_1(x,x) = \frac{1}{2} p_+(x) \ , \qquad (32c)$$

$$L_1(x,x) = -\frac{i}{4}\sinh\frac{u(x)}{2} \ , \tag{32d}$$

were $\sigma_1, \sigma_2, \sigma_3$, denote the standard Pauli matrices.

We can see that the solution for $\mathcal{K}$ and L exists and is unique (except at the singularities of $u(x)$).

In order to derive the inverse scattering equations we consider the following integral in the complex ζ plane (for ζ below C):

$$\int_C \frac{d\zeta'\phi(x,\zeta')e^{ik(\zeta')x}}{(\zeta'-\zeta)a(\zeta')} \ , \tag{33}$$

where C denotes the contour extending from $-\infty + i0^+$ to 0^-, then from 0^+ to $+\infty + i0^+$, and passing over all zeros of $a(\zeta')$.

For u on compact support the Jost functions as well as $a(\zeta)$, $b(\zeta)$ are analytic everywhere (except for $\zeta = 0$ and $\zeta = \infty$), and we can express $\phi(x,\zeta')$ in terms of $\psi(x,\zeta')$ and $\bar{\psi}(x,\zeta')$

$$\int_C \frac{d\zeta'\phi(x,\zeta')e^{ik(\zeta')x}}{(\zeta'-\zeta)a(\zeta')} = \int_C \frac{d\zeta'}{\zeta'-\zeta}\overline{\psi}(x,\zeta')e^{ik(\zeta')x}$$
$$+ \int_C \frac{d\zeta'}{\zeta'-\zeta}\rho(\zeta')\psi(x,\zeta')e^{ik(\zeta')x} \ , \tag{34}$$

where $\rho(\zeta) = b(\zeta)/a(\zeta)$.

Using (21) and applying the Cauchy theorem we find

$$\overline{\psi}(x,\zeta)e^{ik(\zeta)x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2\pi i}\int_C \frac{d\zeta'}{\zeta'-\zeta}\rho(\zeta')\psi(x,\zeta')e^{ik(\zeta')x} \ . \tag{35}$$

If u is not on compact support, the above contour integral can be replaced by an integral along the real axis plus all contributions from the poles of $\rho(\zeta)$.

Note, that $\overline{\psi}(x,\zeta)$ can be also expressed in terms of $\mathcal{K}$ and L, by using (11) and (29). Thus, substituting (29) into (35) and taking the Fourier transform we obtain the inverse scattering equations of the Gelfand-Levitan-Marchenko (GLM) type for $z > x$:

$$i\sigma_2\mathcal{K}(x,z) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F^{(0)}(x+z) +$$

$$\int_x^\infty ds \left[\mathcal{K}(x,s)F^{(0)}(s+z) + \mathcal{L}(x,s)F^{(1)}(s+z)\right] = 0 \ , \qquad (36a)$$

$$i\sigma_2\mathcal{L}(x,z) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F^{(1)}(x+z) +$$

$$\int_x^\infty ds \left[\mathcal{K}(x,s)F^{(1)}(s+z) + \mathcal{L}(x,s)F^{(2)}(s+z)\right] = 0 \ , \qquad (36b)$$

where

$$\mathcal{L}(x,s) = M(x)L(x,s) \ , \qquad (37)$$

$$F^{(n)}(z) = \frac{1}{4\pi}\int_C d\zeta \left(\frac{i}{2\zeta}\right)^n \rho(\zeta)e^{ik(\zeta)z} \ . \qquad (38)$$

Once $\mathcal{K}$ and $\mathcal{L}$ are found from (36a,b), the solution u and its y-derivative u_y can be recovered using (32c,d) and (37)

$$\tanh\frac{u}{2} = \frac{i\mathcal{L}_1(x,x)}{\frac{1}{4} + \mathcal{L}_2(x,x)} \ , \qquad (39a)$$

$$u_y = 8\mathcal{K}_1(x,x) - iu_x \ . \qquad (39b)$$

5 An Example

According to the above, if the solution u vanishes as $x \to \pm\infty$, then we can reconstruct it from the scattering data at any value of y where u and its derivatives are nonsingular for all x. But how do we determine the scattering data? And how do we determine its evolution across any value of y for which u has a singularity? Before we can answer these key questions, let us first study the simplest nontrivial example of a solution u which

(i) is cylindrically symmetric,

(ii) is singular at the origin,

(iii) vanishes sufficiently rapidly as $r \to \infty$.

A closed form of the above solution is not known; however, for $r \to \infty$ we can easily find that the solution approaches the linear limit (5):

$$u \to AK_0(r) \ , \tag{40}$$

where $K_0(r)$ denotes the modified Bessel function and A is a real constant.

If u, u_x and u_y are infinitesimal, then the scattering coefficients can be calculated as standard (linear) Fourier transforms [9,11] of u and its derivatives. In particular, from (18a,b) we find for $u \ll 1$

$$a(y,\zeta) \to 1 \ , \tag{41a}$$

$$b(y,\zeta) \to \int_{-\infty}^{+\infty} \left[p_+(x,y) - \frac{u(x,y)}{8\zeta} \right] e^{-2ik(\zeta)x} dx \ . \tag{41b}$$

Expressing u and p_+ in Cartesian coordinates for $r \to \infty$ we have

$$u(x,y) = AK_0\left(\sqrt{x^2+y^2}\right) \ , \tag{42a}$$

$$u_x(x,y) = -A\frac{x}{\sqrt{x^2+y^2}}K_1\left(\sqrt{x^2+y^2}\right) \ , \tag{42b}$$

$$u_y(x,y) = -A\frac{y}{\sqrt{x^2+y^2}}K_1\left(\sqrt{x^2+y^2}\right) \ . \tag{42c}$$

The Fourier transforms of the above functions can be calculated by using the method of Ref.[12]. For k real and $y > 0$ we find

$$\int_{-\infty}^{+\infty} K_0\left(\sqrt{x^2+y^2}\right)e^{-2ikx}dx = \frac{\pi}{\sqrt{1+(2k)^2}}e^{-y\sqrt{1+(2k)^2}} \ , \tag{43a}$$

$$\int_{-\infty}^{+\infty} \frac{K_1\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}e^{-2ikx}dx = \frac{\pi}{y}e^{-y\sqrt{1+(2k)^2}} \ , \tag{43b}$$

where $\sqrt{1+(2k)^2}$ is assumed to be positive.

Substituting (42) into (41b) and using (43a,b) we obtain for $y > 0$, ζ real and positive ($\xi > 0$)

$$b(y,\zeta) = -A\frac{\pi}{2}e^{-y\sqrt{1+(2k)^2}} \ , \tag{44a}$$

while

$$\bar{b}(y,\zeta) = 0 \ . \tag{44b}$$

Since the scattering data are not analytic for $y = 0$ or for $\zeta = 0$, we should consider separately the case $y > 0$ and $y < 0$ as well as $\xi > 0$ and $\xi < 0$. For $y > 0$ and ζ on the negative real axis ($\xi < 0$) we have

$$b(y,\zeta) = 0 \ , \tag{45a}$$

$$\overline{b}(y,\zeta) = -A\frac{\pi}{2}e^{-y\sqrt{1+(2k)^2}} \ . \tag{45b}$$

Similarly we can find for $y < 0$ and $\xi > 0$:

$$b(y,\zeta) = 0 \ , \tag{46a}$$

$$\overline{b}(y,\zeta) = A\frac{\pi}{2}e^{y\sqrt{1+(2k)^2}} \ , \tag{46b}$$

while for $y < 0$ and $\xi < 0$

$$b(y,\zeta) = A\frac{\pi}{2}e^{y\sqrt{1+(2k)^2}} \ , \tag{47a}$$

$$\overline{b}(y,\zeta) = 0 \ . \tag{47b}$$

On the other hand, it follows from (28,c) and (41a) that

$$a(y,\zeta) = \overline{a}(y,\zeta) = 1 \tag{48}$$

for any $y \neq 0$.

Note, that the expressions (44)-(48) for the scattering data are in agreement with both (28) and (15a,b),(17a,b). Moreover, since we can make u as small as we wish by letting $y \to \pm\infty$, we can conclude that the above results are <u>exact</u> and valid for arbitrary y (except for $y = 0$).

Having determined the scattering data, we are in a position to consider the inverse scattering problem. Since $a(y,\zeta) = 1$, there are no bound states, and contour integral in (38) can be replaced by an integral along the real axis:

$$F^{(n)}(z) = \frac{1}{4\pi}\int_{-\infty}^{+\infty} d\xi \left(\frac{i}{2\xi}\right)^n \rho(\xi)e^{ik(\xi)z} \ , \tag{49}$$

where $\rho(\xi) = b(\xi)/a(\xi)$ and $k(\xi) = \frac{\xi}{2} - \frac{1}{8\xi}$.

Substituting (44a), (45a) and (48) into (49) and performing integration we find for $y > 0$:

$$F^{(0)}(z) = -\frac{A}{8}\left(y + i\frac{z}{2}\right)\frac{K_1\left(\sqrt{\left(\frac{z}{2}\right)^2 + y^2}\right)}{\sqrt{\left(\frac{z}{2}\right)^2 + y^2}} \ , \tag{50a}$$

$$F^{(1)}(z) = -\frac{iA}{8}K_0\left(\sqrt{\left(\frac{z}{2}\right)^2 + y^2}\right) \ , \tag{50b}$$

$$F^{(2)}(z) = \frac{A}{8}\left(y - i\frac{z}{2}\right)\frac{K_1\left(\sqrt{\left(\frac{z}{2}\right)^2 + y^2}\right)}{\sqrt{\left(\frac{z}{2}\right)^2 + y^2}} \ . \tag{50c}$$

For $y \to \infty$, $\rho(\xi)$ becomes infinitesimal and we find from GLM equations (36a,b):

$$u \cong 8i\mathcal{L}_1(x,x) \cong 8iF^{(1)}(2x) = AK_0\left(\sqrt{x^2 + y^2}\right) \ , \tag{51a}$$

$$\begin{aligned} u_y &= 8\mathcal{K}_1(x,x) - iu_x \cong 8F^{(0)}(2x) - iu_x \\ &= -Ay\frac{K_1\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}} \ . \end{aligned} \tag{51b}$$

On the other hand, for $y \to 0$ we note that the limit

$$\lim_{y \to 0^+} F^{(n)}(z)$$

exists. Moreover, similar calculations performed for $y < 0$ show that

$$\lim_{y\to 0^-} F^{(n)}(z) = \lim_{y\to 0^+} F^{(n)}(z) \ , \tag{52}$$

in spite of the fact that the scattering data are discontinuous for $y = 0$.

Thus, for $y = 0$, we have simply

$$F^{(0)}(z) = CK_1(\frac{z}{2}) \ , \tag{53a}$$

$$F^{(1)}(z) = CK_0(\frac{z}{2}) \ , \tag{53b}$$

$$F^{(2)}(z) = CK_1\left(\frac{z}{2}\right) \ , \tag{53c}$$

where $C = -iA/8$, and it remains only a technical problem of solving the GLM equations (36a,b) with relatively simple expressions (53a,b,c) for $F^{(n)}(z)$.

6 Conclusions

In this paper we have discussed the direct and inverse problem associated with the elliptic Sinh-Gordon equation. In particular, attention has been paid to the simplest case of a cylindrically symmetric singular solution, for which we have been able to derive exact analytical expressions for the scattering data (44)-(48). The last step (i.e. solving the GLM equations) cannot be done analytically, however due to simple (and exact) expressions for the kernels $F^{(n)}(z)$ the problem is tractable by numerical means and allows us to find effectively the potential u and its derivatives u_x, u_y.

Generally speaking, we have found the IST approach to be surprisingly effective, in spite of a rather nonstandard application to the elliptic problem. However, some questions remain open:

(i) Both numerical results and the WKB analysis [13] show that the amplitude A (see eq. (40)) is bounded by $A_{max} = 4/\pi$. For $A > A_{max}$ the solution seems to enter a new class which has point singularity surrounded by a singular ring.

(ii) Generalization to the case of two (or many) charges is nontrivial. Preliminary results indicate that the apparent asymptotic charge distribution differs significantly from that of the single charge.

We will address the above problems in a future publication.

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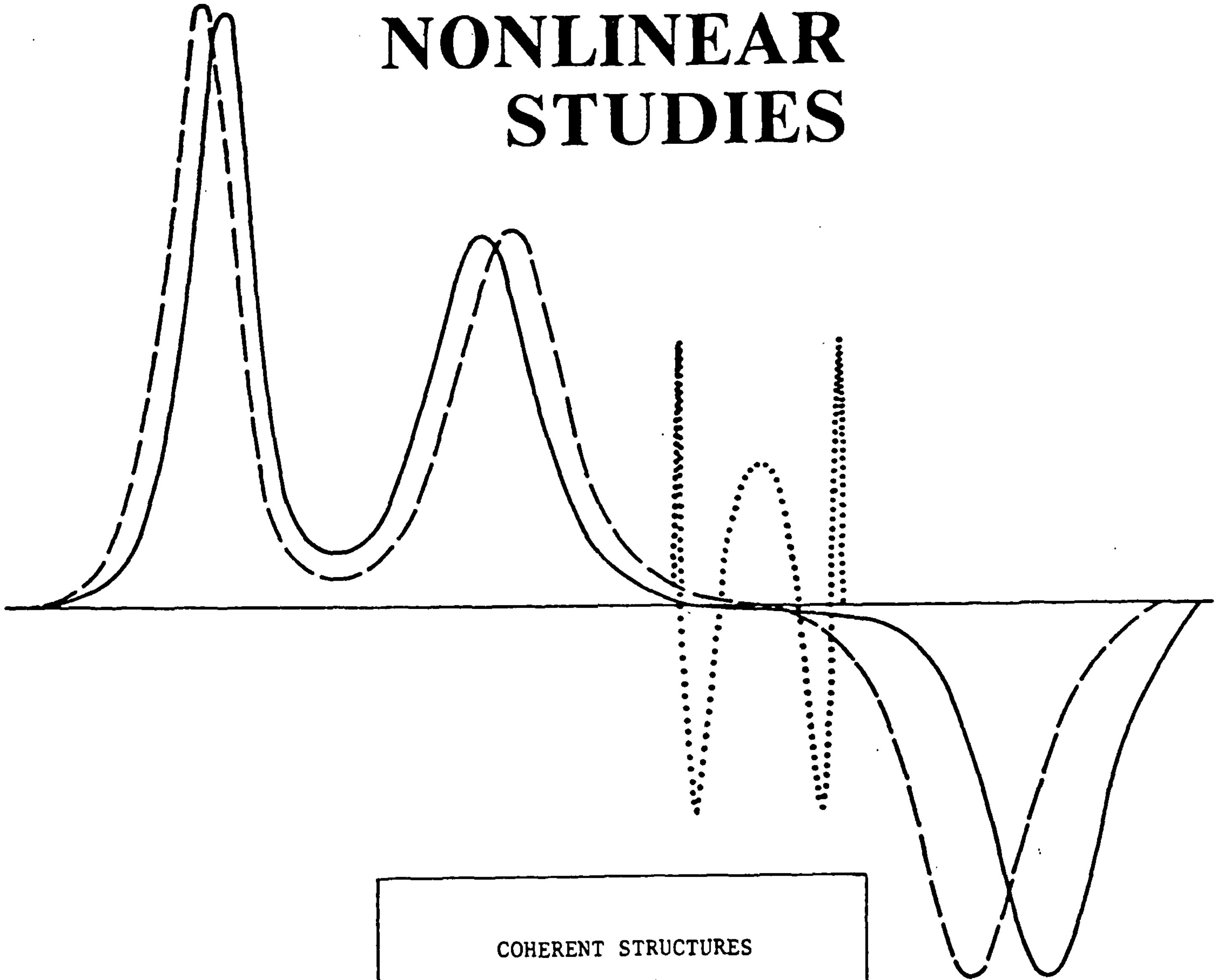

COHERENT STRUCTURES IN THE PLANAR MAGNETRON

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Abstract

An analysis of the initial value problem for the planar magnetron reveals that a coherent structure (convective cell) can be created from arbitrary initial conditions and that these structures will grown linearly in time. This is proposed as the explanation of the origin cf the convective cell formation seen in recent numerical simula

1 Introduction

Although there has been a very long history of the study of the eigenmodel structure of a planar magnetron[1-5] as well as also cylindrical magnetrons[6-8], to our knowledge, no study has been made of the general initial value problem for infinitestimal perturbations for these devices. We do find that the general features of such a treatment for low density nonneutral plasmas has been discussed[9-11] and is well known. In the low density case, it is found that any infinitestimal disturbance in the potential will tend to decay algebraically in time[9-11].

These results have been frequently quoted by others[12] as sufficient reason for ignoring the continuous spectrum also in the high density region of these devices. While that may be true for the electrostatic potential, we shall demonstrate here with a simple asymptotic expansion that such is not so for the fluid motion. Indeed we find that a very important generic algebraic instability always exists in these devices. And it can also occur at low densities, too, although reduced in magnitude by a factor of ω_p^2/Ω^2. Furthermore, this result will demonstrate that "coherent structures[13]" can arise from the plasma fluid equations. The generic features which allow them to exist in this case are the shear flow and the wave-particle resonance.

The simplest example of a solution of a general initial value problem for infinitestimal perturbations of a fluid flow problem has been given by Case[14,15]. As to details, we refer the reader to Refs. [9-11,14,15]. Here we shall briefly outline the solution of the initial value problem for perturbation of the planar magnetron (cylindrical case will be essentially the same except for changes due to the cylindrical coordinates and curvature effects), detailing only the important differences from the standard[9-11,14,15] case.

2 The Initial Value Problem for Infintestimal Perturbations of a Planar Magnetron

We shall model the planar magnetron with the cold-fluid plasma equations using smooth bore boundary conditions. First we shall outline the starting equations and geometry, and then derive the equations relevant to the initial value problem for infinitestinal perturbations.

The cold-fluid equations (with pressure 0) describing the nonrelativistic flow of a non-neutral pure electron plasma are

$$\partial_t n + \vec{\nabla} \cdot (n\vec{v}) = 0, \tag{1a}$$

$$(\partial_t - \vec{v} \cdot \vec{\nabla})\vec{v} + \vec{\mathcal{E}} + \vec{v} \times \vec{\Omega} = 0, \tag{1b}$$

$$\vec{\mathcal{E}} = -\vec{\nabla}\phi, \tag{1c}$$

$$\nabla^2 \phi = \omega_p^2. \tag{1d}$$

Here, n, m, and v denote the electron number density, the electron mass, and the velocity, respectively. The plasma frequency is $\omega_p^2 = 4\pi e^2/m$. The equilibrium we consider corresponds to the planar configuration of Fig. 1 with crossed electric and magnetic fields. The cathode is at $y = 0$, and the anode at $y = \ell$. We define the normalized electric field $\vec{\mathcal{E}_0} = e\vec{\mathbf{E}}_0/m = -eE_0\hat{y}/m$ and the gyrofrequency $\vec{\Omega} = -e(B_0/mc)\hat{z}$.

In this paper we consider electrostatic modes where the magnetic field remains equal to the equilibrium or zeroth-order value at all orders. Hence we will omit the subscript 0 on the Ω even though this is a zeroth-order quantity. We shall also do the same for the plasma frequency shortly. We consider a two-dimensional diode structure, with translational invariance in the z direction both in the equilibrium and for the perturbation quantities. In addition, we assume translational invariance in the x direction for the equilibrium quantities. From Poisson's equation, we have

$$\partial_y \mathcal{E}_0 = \omega_p^2 \tag{2a}$$

where now $\omega_p^2 = 4\pi n_0 e^2/m$, and n_0 is the equilibrium electron density. The electron density n_0 may be specified to be of any form and, hence, is arbitrary. For n_0 or ω_p^2 a monotonic decreasing function of y, the linear diocotron mode is stable for a cold massless low-density electron plasma.[16] This criterion has recently been generalized to warm plasmas without assuming the guiding-center approximation.[17] The equilibrium electron flow velocity is

$$\vec{v}_0 = v_0\hat{x} = \left[\frac{\mathcal{E}_0}{\Omega}\right]\hat{x} \tag{2b}$$

Equations (2a) and (2b) imply that v_0 increases monotonically with y, and thus we have a nonzero velocity shear in the equilibrium.

We next consider first-order perturbations of Eqs. (1). In general, all physical quantities χ may be written as

$$\chi = \chi_0 + \epsilon\chi_1 + \epsilon^2\chi_2 + \cdots, \tag{3a}$$

where ϵ is a measure of the (small) deviation from the equilibrium value χ_0. Here, χ may denote the density, electric field, or velocity. In first order, all quantities are expanded as

$$\chi_1 = \tilde{\chi}_1(y,t)e^{ikx} + c.c. \tag{3b}$$

In this case, Eqs. (1) reduce to

$$D\tilde{n}_1 + ikn_0\tilde{v}_{1x} + \partial_y(n_0\tilde{v}_{1y}) = 0 \tag{4a}$$

$$D\tilde{v}_{1x} - \Delta^2\tilde{v}_{1y}/\Omega - ik\tilde{\phi}_1 = 0 \tag{4b}$$

$$D\tilde{v}_{1y} + \Omega\tilde{v}_{1x} - \partial_y\tilde{\phi}_1 = 0 \tag{4c}$$

$$(\partial_y^2 - k^2)\tilde{\phi}_1 - \tilde{n}_1\left(\frac{\omega_p^2}{n_0}\right) = 0 \tag{4d}$$

where

$$D = \partial_t + ikv_0 \tag{5}$$

$$\Delta^2 = \Omega^2 - \omega_p^2 \tag{6}$$

One can simplify the final equations by using Lagrangian displacements. If we define

$$\tilde{v}_{1x} = D\tilde{\xi}_x - \frac{\omega_p^2}{\Omega}\tilde{\xi}_y \tag{7a}$$

$$\tilde{v}_{1y} = D\tilde{\xi}_y \tag{7b}$$

$$\tilde{n}_1 = -ikn_0\tilde{\xi}_x - \partial_y(n_0\tilde{\xi}_y) \tag{7c}$$

then (4a) is identically satisfied and the remaining three equations become

$$D D \tilde{\xi_x} - \Omega D \tilde{\xi_y} - ik\tilde{\phi_1} = 0 \tag{8a}$$

$$D D \tilde{\xi_y} + \Omega D \tilde{\xi_x} - \omega_p^2 \tilde{\xi_y} - \partial_y \tilde{\phi_1} = 0 \tag{8b}$$

$$(\partial_y^2 - k^2)\tilde{\phi_1} + ik\omega_p^2 \tilde{\xi_x} + \partial_y(\omega_p^2 \tilde{\xi_y}) = 0 \tag{8c}$$

The smooth bore boundary conditions are

$$\tilde{\phi}_1(0,t) = 0 = \tilde{\phi}_1(\ell,t) \tag{9}$$

which arises from the vanishing of the parallel electric field at the cathode and the anode.

Given the value of $\tilde{\xi}_x(y,t=0)$, $\tilde{\xi}_y(y,t=0)$, $\tilde{v_{1x}}(y,t=0)$, and $\tilde{v_{1y}}(y,t=0)$, we wish to construct the solution of (8) for all $t > 0$. This is the initial value problem we shall now solve.

If we define the Laplace transform of a variable as

$$\Xi_x(y,s) \equiv \int_0^\infty e^{-st}\tilde{\xi}_x(y,t)dt \tag{10}$$

by a capital symbol, then eqs (8) becomes

$$p^2\Xi_x - \Omega p \Xi_y - ik\Phi = F_x \tag{11a}$$

$$p^2\Xi_y + \Omega p \Xi_x - \omega_p^2 \Xi_y - \partial_y \Phi = F_y \tag{11b}$$

$$(\partial_y^2 - k^2)\Phi + ik\omega_p^2 \Xi_x + \partial_y(\omega_p^2 \Xi_y) = 0 \tag{11c}$$

where Φ is the Laplace transform of $\tilde{\phi}_1$,

$$p = s + ikv_0(y) \tag{12}$$

and the inhomogeneous terms in (11) are

$$F_x = p\tilde{\xi}_x(t=0) + \tilde{v}_{1x}(t=0) - \frac{\Delta^2}{\Omega}\tilde{\xi}_y(t=0) \tag{13a}$$

$$F_y = p\tilde{\xi}_y(t=0) + \tilde{v}_{1y}(t=0) + \Omega\tilde{\xi}_y(t=0) \tag{13b}$$

By eliminating Φ from (11), one can reduce (11) to the two equations

$$\frac{1}{p^2}\partial_y(p^2\Xi_x) - ik\left(1 - 2\frac{\omega_p^2}{A}\right)\Xi_y = \frac{\Omega}{Ap}(\nabla \cdot F) - \frac{1}{A}(\nabla \times F)_z \tag{14a}$$

$$\frac{1}{A}\partial_y(A\Xi_y) + ik\Xi_x = \frac{1}{A}(\nabla \cdot F) + \frac{\Omega}{pA}(\nabla \times F)_z \tag{14b}$$

where

$$A = p^2 + \Omega^2 \tag{15}$$

$$(\nabla \cdot F) = ikF_x + \partial_y F_y \tag{16a}$$

$$(\nabla \times F)_z = ikF_y - \partial_y F_x \tag{16b}$$

One can now solve the initial problem by solving (14) for $\Xi_x(y,s,k)$ and $\Xi_y(y,s,k)$ subject to the boundary conditions (9). Through (11a), these boundary conditions will relate Ξ_x and Ξ_y at y = 0 and at $y = \ell$. This solution is

$$(p^2\Xi_x - \Omega p\Xi_y - F_x)|_{y=0,\ell} = 0 \tag{17}$$

Given the initial data, the solutions of (14), which are inhomogeneous equations, will be unique except possibly at certain discrete values of s, which are the eigenvalues. At these eigenvalues of s, the function Ξ_x and Ξ_y satisfy the homogeneous part of eqs (14). In actuality, the solution of the initial value problem will give Ξ_x and Ξ_y as having a simple pole in s (for all y) as s approaches any one of these eigenvalues[14,15].

But the solution of (14) also has other singularities in s. These are y dependent singularities and occur at the singular points of the second-order differential system (14). From (14), it is obvious that the points $p = 0$ and $A = 0$ are ordinary singular points for the differential system (14). The position of these singularities depend on s through (12) and (15). Since we shall have need of them later, we shall give the form of the solution near these singularities.

Near $p = 0$, and as $p \to 0$, except for an overall multiplicative constant (in y, but possibly dependent on s and k) the various Laplace transforms will approach

$$\Xi_x \to \frac{ik\omega_p^4}{\Omega^2 p^2} + \cdots \tag{18a}$$

$$\Xi_y \to \frac{ik\omega_p^2}{\Omega p} + \cdots \tag{18b}$$

$$V_x \to -\frac{(\partial_y \omega_p^2)}{\Omega} \ln p + \cdots \tag{18c}$$

$$V_y \to \frac{ik\omega_p^2}{\Omega} + \cdots \tag{18d}$$

$$ik\Phi \to -\frac{ik\Delta^2\omega_p^2}{\Omega^2} + \cdots \tag{18e}$$

$$\partial_y\Phi \to -(\partial_y\omega_p^2)\ln p + \cdots \tag{18f}$$

$$N \to -ikn_0\frac{\partial_y\omega_p^2}{\Omega p} + \cdots \tag{18g}$$

where $V_x(V_y, N)$ is the Laplace transform of $\tilde{v}_{1x}(\tilde{v}_{1y}, \tilde{n}_1)$.

Similarly near $A \to 0$, we have

$$\Xi_x \to \frac{2\Omega^2}{A} + \cdots \tag{19a}$$

$$\Xi_y \to \frac{2p\Omega}{A} + \cdots \tag{19b}$$

$$V_x \to \frac{2p\Delta^2}{A} + \cdots \tag{19c}$$

$$V_y \to -\frac{2\Omega^3}{A} + \cdots \tag{19d}$$

$$ik\Phi \to -\Omega^2\ln A + \cdots \tag{19e}$$

$$\partial_y\Phi \to -\frac{2p\Omega\omega_p^2}{A} + \cdots \tag{19f}$$

$$N \to -4ikn_0\frac{\omega_p^2\Omega^2}{A^2} + \cdots \tag{19g}$$

Now, every line in eqs (18) and (19) have also logarithmic terms in higher order, but near the singularity, they only dominate in (18c) and (19e). Nevertheless, their existance means that in addition to the pole singularities discussed above, the solutions of (14) must also have branch cuts in the complex s-plane. These cuts will be located at all values of s for which either p or A will vanish for some value of y between the cathode and anode. From (12) and (15), one can see that these branch cuts would lie along the imaginary s-axis, and in general would have three sections as shown in Fig 2. The quantity v_f is the value of $v_0(y)$ at the anode. If $kv_f > \Omega$, then the three sections would merge into one branch cut.

Now that we have described the Laplace transform of the general solution, let us next look at the asymptotic form of the general solution.

3 The Large t Limit of the Initial Value Problem

We shall now obtain the large time asymptotic solution of the initial value problem. Given the Laplace Transform, say $\Xi_x(y, s, k)$, then the original function, $\tilde{\xi}_x(y, t, k)$ is constructed from

$$\tilde{\xi}_x(y, t, k) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} \Xi_x(y, s, k) ds \tag{20}$$

where the contour is to be taken to the right of all singularities in the complex s-plane. The functions are all analytic for large s and also vanish as $| s | \rightarrow \infty$. Therefore we may close the contour to the left of the imaginary axis, and shrink it down into individual contours around all branch points and other singularities. These final contours are indicated in Fig 2.

There are three basic contributions to the solution of the initial value problem: i) contours around the eigenvalues, ii) contours around the branch cuts, and iii) inside each branch cut, for any fixed value of y, there will be a regular singular point of the differential system (14). These latter singularities are tabulated in (18) and (19).

The contribution from the eigenvalues will be a (global) eigenmode with a frequency of $Im(s_j)$ and a growth rate of $Re(s_j)$ where s_j is the eigenvalue. This contribution has been very well described in the literature[1-8]. The contributions from the branch cuts (excluding all pole singularities inside the cut) will always decay at least algebraically in time[9-11,14,15], usually at least as fast as t^{-1}. Thus it is only a transient part of the solution.

On the other hand, the pole singularities inside the branch cuts can give a larger contribution. For example, (18a) has a pole or order 2 about $p = 0$. Inserting such a pole into (20) will give a growing contribution. Let $\alpha(s, y)$ be analytic in s near $p = 0$. Then one may easily show that for large times

$$\frac{1}{2\pi i}\oint \frac{\alpha(s,y)}{p^2(s,y)}e^{st}ds = te^{-ikv_0t}\alpha(-ikv_0, y) + \cdots \tag{21}$$

Similarly

$$\frac{1}{2\pi i}\oint \frac{\alpha(s,y)}{p(s,y)}e^{st}ds = e^{-ikv_0t}\alpha(-ikv_0, y) + \cdots \tag{22}$$

$$\frac{1}{2\pi i}\oint \alpha(s,y)e^{st}(\ln p)ds \to 0 \tag{23}$$

From (21)-(23), one can easily read off the asymptotic values of the various quantities in (18) and (19). For (18), near $p = 0$, we see that for large t

$$\tilde{\xi}_{1x} \to ikC_0t\frac{\omega_p^4}{\Omega^2}e^{-ikv_0t} + \cdots \tag{24a}$$

$$\tilde{\xi}_{1y} \to ikC_0 t \frac{\omega_p^2}{\Omega} e^{-ikv_0 t} + \cdots \tag{24b}$$

$$\tilde{v}_{1x}, \tilde{v}_{1y}, ik\tilde{\phi}_1, \partial_y \tilde{\phi}_1 \to 0 \tag{24c}$$

$$\frac{\tilde{n}_1}{n_0} \to -ikC_0 \left(\frac{\omega_p^2}{\Omega}\right) \left(\frac{\partial_y n_0}{n_0}\right) e^{-ikv_0 t} + \cdots \tag{24d}$$

where C_0 is the overall normalization constant, $C_0(s,k)$, for (18), evaluated at $p = 0$. This constant is determined by the initial data, and in general is nonzero.

As (24) shows, the solution is rather simple. For large t, the y displacements approach a constant amplitude and are carried along with the background flow, v_0. (Note that when the $e^{-ikv_0 t}$ term is combined with the e^{ikx} term in (3b), then $\tilde{\xi}_{1x}, \tilde{\xi}_{1y}$, and $\tilde{n}_1$ will be functions of the combination $(x - v_0 t)$). Of course, any displacements in the y-direction shifts the particle into a different background flow. Whence there is a lateral displacement of the fluid particle away from its original position which grows linearly in time. This lateral displacement is also directly proportional to the background velocity difference between the initial and final layer. This gives $\xi_x = t\xi_y \partial_y v_0$ and is exactly the result found by (24a). At the same time, the density shifts according to the vertical displacement of the particles. Such motion is well known from the fluid dynamics of shear flows[18]. Nothing more needs to be said about it here except that it verifies that the treatment so far is correct.

Now consider the solution at the so-called "magnetron instability[1,2,12,19]", which is where $A = 0$, or

$$p = \pm i\Omega \tag{25}$$

As before, we obtain

$$\tilde{\xi}_{1x} \rightarrow \mp i C_{\pm} \Omega e^{\pm i \Omega t} e^{-ikv_0 t} \tag{26a}$$

$$\tilde{\xi}_{1y} \rightarrow C_{\pm} \Omega e^{\pm i \Omega t} e^{-ikv_0 t} \tag{26b}$$

$$\tilde{v}_{1x} \rightarrow C_{\pm} \Delta^2 e^{\pm i \Omega t} e^{-ikv_0 t} \tag{26c}$$

$$\tilde{v}_{1y} \rightarrow \pm i C_{\pm} \Omega^2 e^{\pm i \Omega t} e^{-ikv_0 t} \tag{26d}$$

$$ik\tilde{\phi}_1 \rightarrow 0 \tag{26e}$$

$$\partial_y \tilde{\phi}_1 \rightarrow -C_{\pm} \Omega \omega_p^2 e^{\pm i \Omega t} e^{-ikv_0 t} \tag{26f}$$

$$\frac{\tilde{n}_1}{n_0} \rightarrow ikC_{\pm} \omega_p^2 t e^{\pm i \Omega t} e^{-ikv_0 t} \tag{26g}$$

where $C_{\pm}(s, k)$ are the constants of proportionality for (19).

Now the motion is more complicated. However the interpretation is again simple. Per (26) the fluid particles now undergo cyclotron motion with a radius of $|C_{\pm}\Omega|$. They are also carried along with the background flow. The resulting fluid velocity is assymetrical[20,21] (see also (7)). The electrostatic potential and the parallel electric field both vanish, but the vertical electric field <u>does not vanish</u>. Instead it approaches a constant amplitude proportional to the density, and is exactly equal to $\xi_y \omega_p^2$. As a consequence of this vertical electric field, there is a constant flux of particles <u>into</u> this motion, with the number of particles involved in the motion growing linearly in time.

Note that the electrostatic potential does vanish, but that the fluid velocities do not. As such, it is the fluid motion which is most significant for this solution. The electrical potential has the least importance. It vanishes. The vertical electrical field seems to arise as a result of only the cycling cyclotron particles. When they move up vertically, the disturbed charge density creates a more intense vertical electric field at that point. And the resulting imbalance in the vertical component of the force equation due to the shear flow requires this vertical electric field.

Another way to look at this solution is to consider the effect of electrons undergoing cyclotron motion in a shear flow. Even if they would be initially in phase, due to the shear flow, different layers will go out of phase, creating large vertical density gradients which generate the vertical electric field. And this time oscillating electric field will then pull more particles into a cyclotron motion. Note that the flow is <u>compressible</u> since $\vec{\nabla} \cdot \vec{v} \neq 0$.

This motion seems to describe the recent numerical simulations of a 2 vane magnetron done by MRC[22]. In Fig 3 we show a series from these simulations. What one can observe is a slow growth of a circular structure which extends over two vanes. This structure is actually composed of rotating particles and is called a convective cell[23–25]. The number of particles involved in the motion do seem to grow linear in time. Unfortunately, exact numerics and statistics adequate for a good quantitative comparison are currently unavailable.

However simulation results on a 10 vane magnetron[26] indicated that convective cell formation was less visible and certain. Although one could observe an occasional sporatic formation of vorticies which would seem to slowly grow for some indefinite time, nevertheless they would eventually collide and dissipate away. In any case, their formation was never observed as clearly and as certain as was in the 2 vane simulations.

This is not inconsistant with these results, because in the 2 vane case, only the fundamental k-vector and its harmonics could exist due to the periodicity. Thus one had, in effect, only one value of k present. But in the 10 vane simulations, sidebands of k (0.9k, 1.1k, etc) could simultaniously coexist with the fundamental. One could approximate this situation with an almost continous spectrum of k, where the general solution would be an integral over k. As a consequence of this, there would be an additional dephasing which would reduce the growth[14,15] predicted by (24) and (26). And this is consistant with the simulations of the 10 vane magnetron.

4 The Direct Asymptotic Solution

If the solutions (24) and (26) are true, then one should be able to obtain them directly from (8). Indeed one can. For the $p \to 0$ solution, simply take the ansaltz

$$\tilde{\phi} = e^{-ikv_0 t} \sum_{n=0}^{\infty} \frac{\gamma_n^{(0)}(y,k)}{t^{n+2}} \tag{27a}$$

$$\tilde{n}_1 = e^{-ikv_0 t} \sum_{n=0}^{\infty} \frac{\eta_n^{(0)}(y,k)}{t^n} n_0(y) \tag{27b}$$

$$\tilde{v}_{1x} = e^{-ikv_0 t} \sum_{n=0}^{\infty} \frac{v_n^{(0)}(y,k)}{t^{n+1}} \tag{27c}$$

$$\tilde{v_{1y}} = e^{-ikv_0 t} \sum_{n=0}^{\infty} \frac{u_n^{(0)}(y,k)}{t^{n+2}} \tag{27d}$$

Then eq. (8) provides recursion relations which determines all the functions $\gamma_n^{(0)}, \eta_n^{(0)}, v_n^{(0)}$, and $u_n^{(0)}$ in terms of just one function. Choosing this function to be $\gamma_0^{(0)}$, we find

$$\eta_0^{(0)} = -\frac{k^2\Omega_p^2}{\omega^2}\gamma_0^{(0)} \tag{28a}$$

$$v_0^{(0)} = -ik\frac{\omega_p^2}{\Omega^2}\gamma_0^{(0)} \tag{28b}$$

$$u_0^{(0)} = -\frac{ik}{\Omega}\gamma_0^{(0)} \tag{28c}$$

$$\gamma_1^{(0)} = \frac{2\Omega}{ik\omega_p^2}\partial_y\gamma_0^{(0)} \tag{29a}$$

$$\eta_1^{(0)} = -\frac{ik\gamma_0^{(0)}}{\Omega\omega_p^2}\partial_y\omega_p^2 \tag{29b}$$

$$v_1^{(0)} = -\frac{1}{\Omega}\partial_y\gamma_0^{(0)} \tag{29c}$$

$$u_1^{(0)} = \frac{-\Omega^2}{\Delta^2\omega_p^2}\partial_y\gamma_0^{(0)} \tag{29d}$$

and etc. Note the inverse powers of ω_p^2 in (29a,b,d). These terms indicate that this asymptotic series will require a longer time for the higher order terms to become smaller wherever the density is small. And the relative sizes of these terms indicate the time required for a certain accuracy to be achieved. Comparing (28a) and (24d) reveals that

$$\gamma_0^{(0)} = \frac{\Omega C_0}{ik}\frac{\partial_y n_0}{n_0} \tag{30}$$

Since solving (11) can give one the value of C_0, we see that the asymptotic solution (27) can also be related to the initial data.

Now consider the magnetron instability where $A \rightarrow 0$. The correct ansaltz is

$$\tilde{\phi} = e^{\pm i\Omega t} e^{-ikv_0 t} \sum_{n=0}^{\infty} \frac{\gamma_n^{(\pm)}(y,k)}{t^{n+2}} \tag{31a}$$

$$\tilde{n}_1 = e^{\pm i\Omega t} e^{-ikv_0 t} \sum_{n=0}^{\infty} \frac{\eta_n^{(\pm)}(y,k)}{t^{n-1}} n_0(y) \tag{31b}$$

$$\tilde{v}_{1x} = e^{\pm i\Omega t} e^{-ikv_0 t} \sum_{n=0}^{\infty} \frac{v_n^{(\pm)}(y,k)}{t^{n}} \tag{31c}$$

$$\tilde{v}_{1y} = e^{\pm i\Omega t} e^{-ikv_0 t} \sum_{n=0}^{\infty} \frac{u_n^{(\pm)}(y,k)}{t^{n}} \tag{31d}$$

Again, the lowest order solutions are

$$\eta_0^{(\pm)} = -\frac{k^2\omega_p^2}{\Omega^2}\gamma_0^{(\pm)} \tag{32a}$$

$$v_0^{(\pm)} = ik\frac{\Delta^2}{\Omega^2}\gamma_0^{(\pm)} \tag{32b}$$

$$u_0^{(\pm)} = \mp k\gamma_0^{(\pm)} \tag{32c}$$

$$\gamma_1^{(\pm)} = 0 \tag{33a}$$

$$\eta_1^{(\pm)} = -\frac{ik}{\Omega\omega_p^4}\partial_y\left(\omega_p^4\gamma_0^{(\pm)}\right) \tag{33b}$$

$$v_1^{(\pm)} = \pm\frac{\Omega k}{\omega_p^2}\gamma_0^{(\pm)} - \frac{\Delta^2}{\Omega\omega_p^2}\partial_y\gamma_0^{(\pm)} \tag{33c}$$

$$u_1^{(\pm)} = ik\frac{\Omega}{\omega_p^2}\gamma_0^{(\pm)} \mp i\frac{\Omega}{\omega_p^2}\partial_y\gamma_0^{(\pm)} \tag{33d}$$

and etc. As before, convergence will be poorest in the low density regions. Also from (32a) and (26g)

$$\gamma_0^{(\pm)} = \frac{\Omega^2}{ik}C_\pm \tag{34}$$

Note that in (26), $C_\pm$ is the overall constant of proportionality for the solution (19). Thus, $C_\pm$ can be found from the initial data, and by (34), $\gamma_0^{(\pm)}$ can also be found from the initial data. Thus given the initial data, one can in principle construct the time asymptotic solution by knowing simply three function of y and k; namely $\gamma_0^{(\pm)}(y,k)$ and $\gamma_0^{(0)}(y,k)$.

5 Summary

We have presented a generic algebraic instability in the planar magnetron. It will always be present except for those rare initial data for which the constants $C_\pm$ vanish. It is furthermore independent of the density profile. Thus even if a stable equilibrium density profile[27] existed, this algebraic instability would cause the density profile to drift away from equilibrium.

This instability seems to have been important in the 2 vane magnetron simulations. However, it seemed to be less dramatic when sidebands could be present, as in the 10 vane simulations.

Figure Captions

Fig. 1: Geometry and the shear flow in the planar magnetron.

Fig. 2: The complex s-plane, showing the branch cuts (x - - - x) and the location of possible eigenvalues ($*$) for the Laplace Transforms. At each value of y, there is also a singularity inside each branch cut which is located at $-ikv_0(y)$ below the top of the branch cut. Thus these singularities move as a function of y. v_f is the value of $v_0(y = \ell)$. The contour C is discussed in the text.

Fig. 3: The formation of a convective cell. (Courtesy of Mission Research Corp.)

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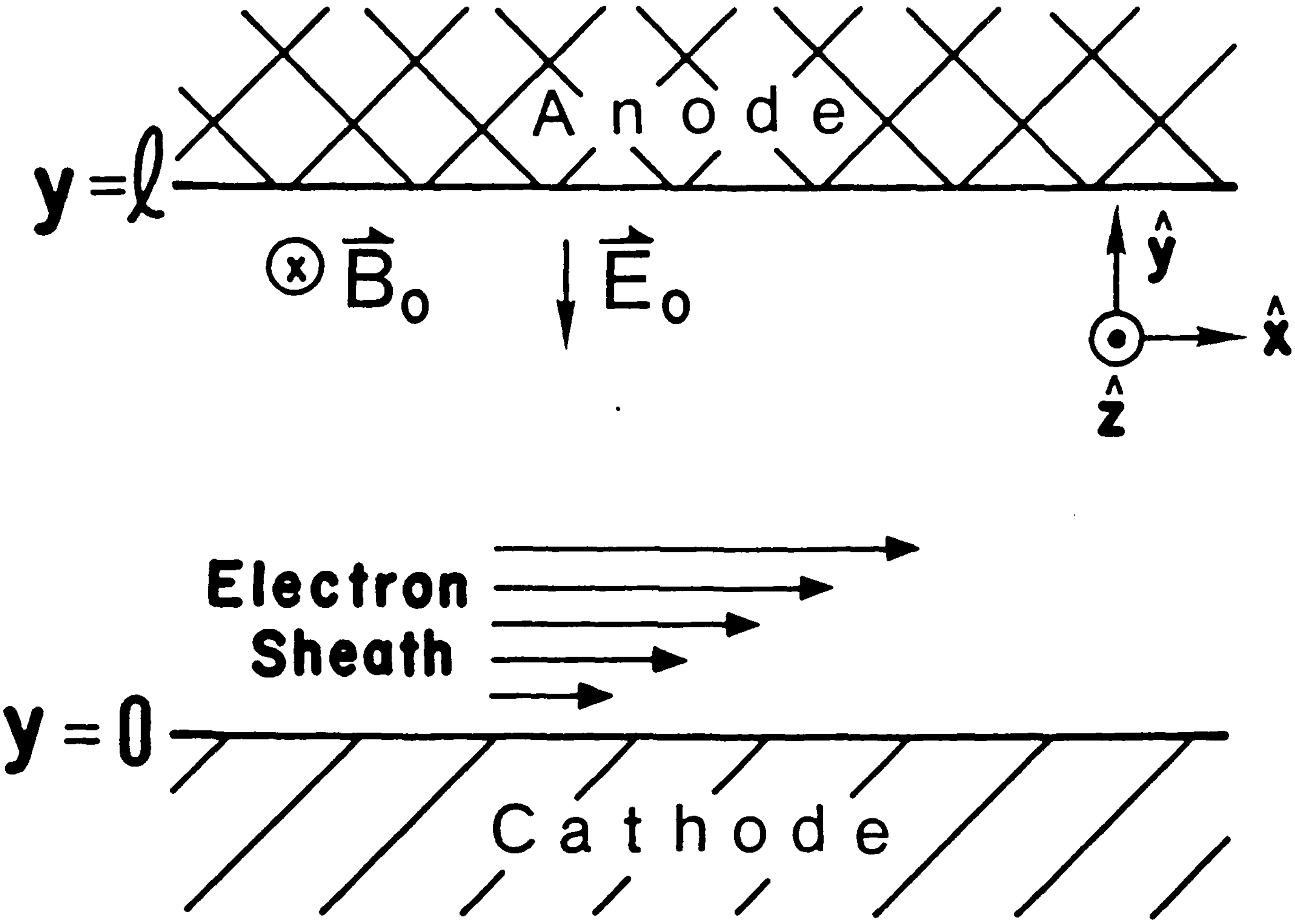

FIG. 1 - Geometry and the shear flow in the planar magnetron

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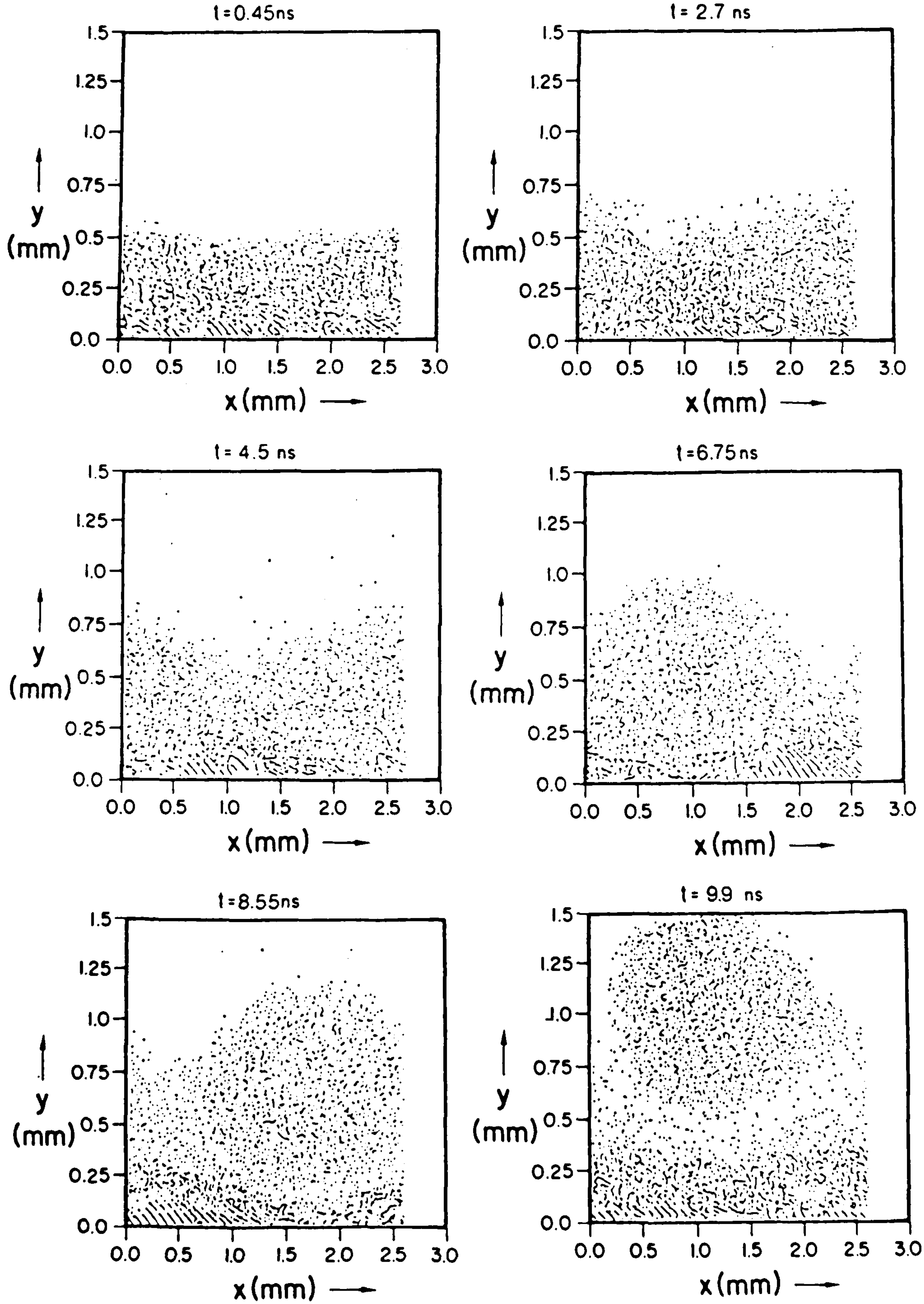

FIG. 3 - The formation of a convective

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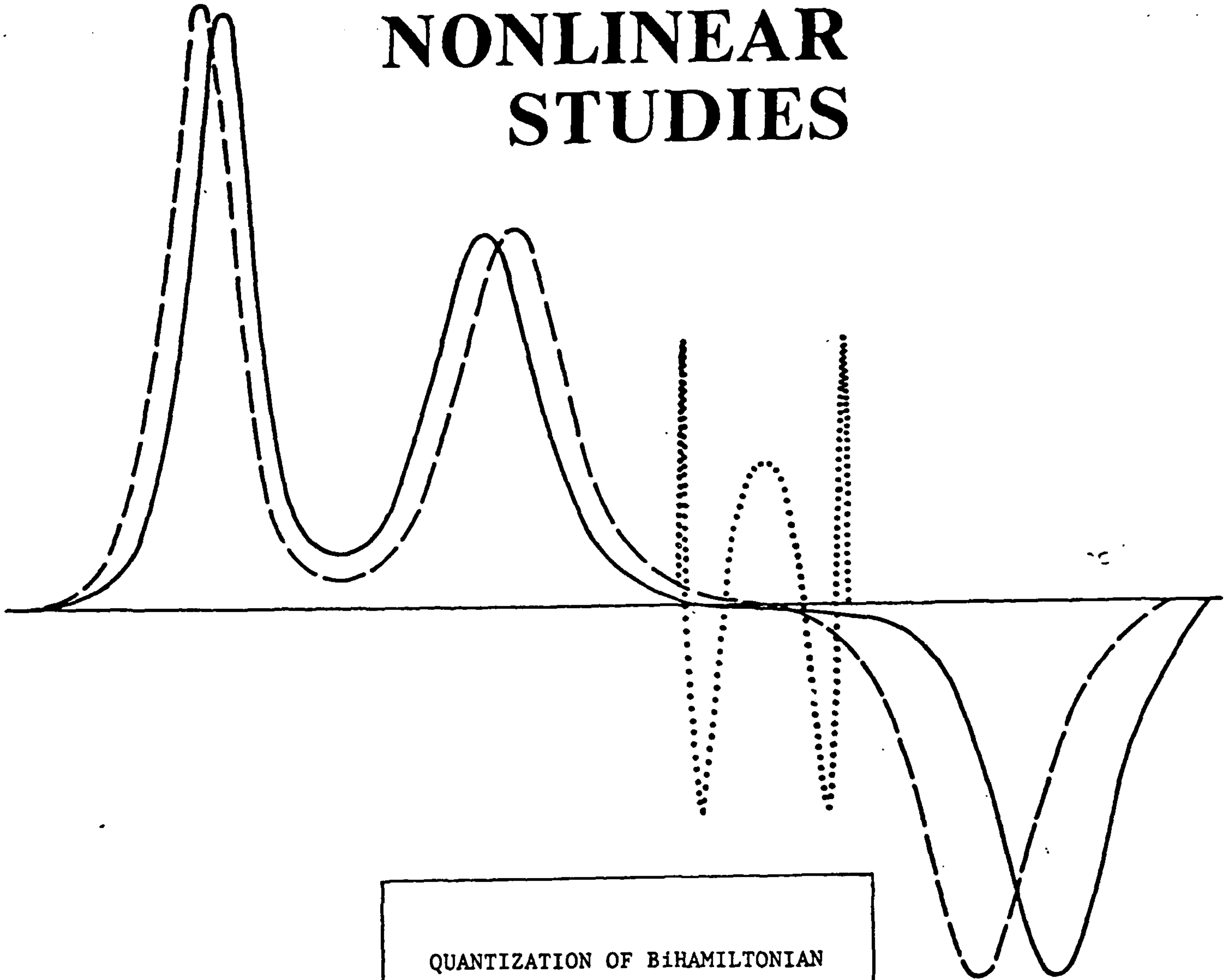

QUANTIZATION OF BiHAMILTONIAN SYSTEMS

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Quantization of BiHamiltonian Systems

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Abstract

One of the distinguishing features of soliton equations is the fact that they can be written in Hamiltonian form in more than one way. Here we compare the different quantized versions of the soliton equations arising in the AKNS inverse scattering scheme. We find that, when expressed in terms of the scattering data, both quantized versions are essentially identical.

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In 1975, one of the present authors[1] showed how to obtain the quantized levels of the nonlinear Schrödinger equation using the action-angle variables (canonical coordinates) of the AKNS scattering data. The symplectic form used to effect the reduction to canonical coordinates was based on the standard Hamiltonian structure for the nonlinear Schrödinger equation. The method used was a nonlinear generalization of one of the standard methods for the second quantization of the electromagnetic field. As presented in the textbook by Schiff[2], one takes the classical electromagnetic field and decomposes it into normal modes (Fourier components). The key idea in this approach is that the classical electromagnetic Hamiltonian will decompose into a sum of noninteracting classical Hamiltonians, each of which has just two degrees of freedom and is easily quantized by itself. This method of quantization bypasses all the inherent difficulties of fully quantizing the system, including the factor-ordering problem, defining the quantum field operators for the fundamental fields, etc.[3] It is fundamentally based on the symmetries of the classical system, and reduces the problem to one of quantizing noninteracting particles[4]. In this way, the original difficult second quantization problem is reduced to a simpler set of noninteracting problems. The advantage of this simpler solution is tremendous when one considers the information that one can glean from it. First, one can obtain the spacings of the energy levels. One also discovers which quantum variables will commute, and which modes will have a particle-like behavior. Of course, for a full quantum theory, one still has to deal with a number of remaining difficult problems, including finding a consistent factor-ordering for the quantum operators, evaluating matrix elements, etc. Unfortunately, the solution to this larger quantization problem may well be multi-valued[3]. However, in the meantime, one has been able to immediately isolate the above-mentioned important features of second quantization, and, very importantly, those quantities which would have the same common solution for every possible consistent second quantization. Thus, any difficulty which would be found at this level would also be present in *any* quantum field theory. And a study by this method can provide valuable insight into the structure of the more thorny parts of the second-quantization problem.

The symplectic form used in ref. 1 to effect the reduction to canonical coordinates was based on the first Hamiltonian structure for the nonlinear Schrödinger equation. In 1978, Magri[5] showed how many soliton equations, including the nonlinear Schrödinger equation, could be written as biHamiltonian systems, meaning that they have two distinct, but compatible, Hamiltonian structures. Indeed, his fundamental result showed that, subject to some technical hypotheses[5,6] any biHamiltonian system is completely integrable in the sense that it has infinitely many conservation laws in involution and corresponding commuting Hamiltonian flows.

From the viewpoint of quantum mechanics, the existence of more than one Hamiltonian structure for a given classical mechanical system raises the possibility of there existing more than one quantized version of this system, even at the level of quantization considered in ref. 1. The resulting ambiguity in the quantization procedure raises serious physical doubts as to the mathematical framework of quantization. However, the main result to be proven here is that, for AKNS soliton equations[7], both quantized versions are essentially the same. We demonstrate that, in terms of the respective canonical coordinates on the scattering data, the two Hamiltonians have identical expressions, and hence identical quantum versions. Indeed, we conjecture that this phenomenon is true in general - *quantization does not depend on the underlying Hamiltonian structure*. (The results of Dodonov et. al.[8], in which an ambiguity in the quantization procedure for certain finite-dimensional biHamiltonian systems is supposedly demonstrated, are erroneous, since they fail to incorporate the important topological properties of phase space properly in their picture. Indeed, their ambiguity is just a version of the ambiguity inherent in the quantization of two-dimensional Hamiltonian systems, which we discuss in detail below.) Moreover, we will see that for the other members of the associated hierarchy of soliton equations the only difference in the quantum versions is in the choice of weighting factor for the quantum operators corresponding to the continuous spectrum, the weight being determined by the classical dispersion relation, and the replacement of the bound state Hamiltonians. Thus, the effect of quantizing different members of the soliton hierarchy will only be significant for the bound states/solitons.

Our presentation relies heavily on the notation and results in earlier papers by Kaup and Newell[1,9,10] on the closure of the squared eigenfunctions for the AKNS scattering problem. The key to our result is the well-known fact that the recursion operator, which is built out of the two Hamiltonian operators for the system[5,6] is essentially the squared eigenfunction operator. Since variations in the potential for the AKNS scattering problem are expressed in terms of the squared eigenfunctions, this means that the second symplectic form can be simply written down in explicit form; in terms of the scattering data, it differs from the first symplectic form only by a weighting factor in the continuous spectrum, and a change in the discrete components. However, the corresponding difference in weighting factors for the two Hamiltonians exactly cancels out the weighting factor for the two symplectic forms, while the discrete components reduce simply to the quantization of a two dimensional Hamiltonian system, based on different symplectic structures. Thus, the entire quantum ambiguity reduces to the simple matter of an ambiguity in the quantization of two dimensional Hamiltonian systems, a problem that is easily handled.

Our notation is as follows. Hamilton's equations are

$$\partial_t Q^\alpha = J^{\alpha\beta}\,\partial_\beta H, \tag{1}$$

where $Q = \{Q^\alpha\}$ are the dynamical variables (the p's and the q's), $J = [J^{\alpha\beta}]$ is the Hamiltonian operator, which determines the underlying Hamiltonian structure of the phase space, and H is the Hamiltonian function or density. For instance, for a harmonic oscillator, one would take

$$Q = \begin{pmatrix} q \\ p \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{and} \qquad H = \frac{1}{2}(p^2 + q^2).$$

When Q is a function of a continuous variable, the sum over the dummy indices in (1) is understood to include the appropriate integration, and the partial derivative is understood to be a functional derivative instead. The Poisson bracket determined by such a Hamiltonian operator has the form

$$\{ F, G \} = (\partial_\alpha F)\, J^{\alpha\beta}\, \partial_\beta G\,, \tag{2}$$

which requires the symplectic two-form to be

$$\Omega = \frac{1}{2}\, dQ^\alpha \wedge J^{-1}_{\alpha\beta}\, dQ^\beta\,. \tag{3}$$

For the harmonic oscillator, this reduces to the familiar canonical form

$$\Omega = dp \wedge dq. \tag{4}$$

Therefore, the operator J needs to be skew adjoint, and satisfy the additional condition that the Poisson bracket (2) satisfy the Jacobi identity, which is equivalent to the requirement that the two form Ω be closed[6].

Before presenting the main results, we discuss a simple but crucial fact that any two dimensional Hamiltonian system has a unique quantized version, even though it has many different Hamiltonian structures. In terms of the standard Hamiltonian structure prescribed by the canonical two-form (4), Hamilton's equations take the classical form[11]

$$p_t = -\frac{\partial H}{\partial q}, \qquad q_t = \frac{\partial H}{\partial p}. \tag{5}$$

In $\mathbb{R}^2$, any nonzero two-form $\lambda(p,q)\, dp \wedge dq$ is always closed, and hence determines a Hamiltonian operator

$$\mathcal{J} = \begin{pmatrix} 0 & -\frac{1}{\lambda} \\ \frac{1}{\lambda} & 0 \end{pmatrix}.$$

It is easy to see that (5) can be written in Hamiltonian form using this second Hamiltonian structure if and only if λ is a function of the Hamiltonian H. In this case, the new Hamiltonian function is

$$H_2(p,q) = \Phi[\, H(p,q)\,],$$

where $\Phi(\xi)$ is any nonvanishing scalar function, and

$$\Omega_2 = \Phi'[\, H(p,q)\,]\, dp \wedge dq \tag{6}$$

is the second symplectic form. Re-expressing Ω_2 in canonical form will lead to new *canonical* variables $\tilde{p}$, $\tilde{q}$, and an ostensibly different quantized version. However, provided this transformation does not affect the phase space topology, it is not hard to see that these two quantized versions will end up being identical, at least in the semi-classical limit, and so there is no ambiguity in the (semi-classical) quantization of two-dimensional Hamiltonian systems.

We now turn to our problem at hand. For simplicity, we will consider the general nonlinear Schrödinger equation

$$i\, q_t = -q_{xx} + 2\, r\, q^2, \tag{7a}$$

$$i\, r_t = r_{xx} + 2\, q\, r^2, \tag{7b}$$

in detail. However, our arguments will work equally well for any other soliton equation associated with the AKNS spectral problem[7]; see the remarks at the end of the paper. For $r = \pm q^*$, (7) reduces to the single equation

$$i\, q_t = -q_{xx} \pm 2\, (q^*\, q)\, q, \tag{8}$$

which is the form of the nonlinear Schrödinger equation in which all physical constants, e.g. $\hbar$, m, etc., have been set equal to 1. According to Magri[5] the nonlinear Schrödinger equation can be written as a biHamiltonian system

$$\psi_t = J_1\,\partial H_1 = J_2\,\partial H_2 . \tag{9}$$

The first Hamiltonian can be identified with the (signed) energy

$$H_1 = \pm E = \int_{-\infty}^{\infty} (q_x r_x + q^2 r^2)\,dx , \tag{10}$$

while the second Hamiltonian is the field momentum

$$H_2 = P = i\int_{-\infty}^{\infty} (r\,q_x - q\,r_x)\,dx . \tag{11}$$

The two Hamiltonian operators are given by

$$J_1 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{12}$$

$$J_2 = \frac{1}{2}\,\sigma_1\,\partial_x + \begin{pmatrix} q\int_{-\infty}^{x} q & -q\int_{-\infty}^{x} r \\ -r\int_{-\infty}^{x} q & r\int_{-\infty}^{x} r \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{13}$$

(In our notation[6] we have omitted the delta functions used by some authors.) Moreover, these Hamiltonian structures are compatible, in the sense that any linear combination $c_1 J_1 + c_2 J_2$ is also Hamiltonian. Therefore, according to the theorem of Magri the operator

$$R = J_2 \cdot J_1^{-1} \tag{14}$$

is a recursion operator for the general nonlinear Schrödinger equation, leading to an infinite hierarchy of mutually commuting biHamiltonian flows.

To determine the two quantized versions of the nonlinear Schrödinger equation, we need to introduce canonical coordinates and momenta, which will be found among the scattering data for the associated eigenvalue problem. We begin by recalling how this was done in ref. 1 for the first symplectic form. The general nonlinear Schrödinger equation can be solved using the AKNS eigenvalue problem[7]

$$v_{1,x} + i\,\zeta\,v_1 = q\,v_2, \qquad v_{2,x} - i\,\zeta\,v_2 = r\,v_1. \tag{15}$$

We let $\phi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be the solution to (15) satisfying the boundary conditions

$$\phi \to e^{-i\zeta x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x \to -\infty, \qquad \phi \to \begin{pmatrix} a(\zeta)\, e^{-i\zeta x} \\ b(\zeta)\, e^{i\zeta x} \end{pmatrix}, \quad x \to \infty,$$

for $\mathrm{Im}\,\zeta > 0$. Similarly, let $\bar{\phi} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be the solution to (15) satisfying the boundary conditions

$$\bar{\phi} \to e^{-i\zeta x} \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad x \to -\infty, \qquad \bar{\phi} \to \begin{pmatrix} \bar{b}(\zeta)\, e^{-i\zeta x} \\ -\bar{a}(\zeta)\, e^{i\zeta x} \end{pmatrix}, \quad x \to \infty,$$

for $\mathrm{Im}\,\zeta < 0$. This serves to define the scattering coefficients $a, b, \bar{a}, \bar{b}$, which also satisfy

$$\bar{a}(\zeta)\, a(\zeta) + \bar{b}(\zeta)\, b(\zeta) = 1. \tag{16}$$

The ratio $\rho(\xi) = b(\xi)/a(\xi)$, ξ real, serves to define the continuous spectrum of the scattering data for (4). The zeros of $a(\zeta)$ in the upper half plane correspond to the bound states, and are denoted as $\zeta_j = \xi_j + i\,\eta_j$, $j = 1,\ldots,N$. Finally let b_j denote the value of b at ζ_j, and let ρ_j denote the residue of ρ at the pole ζ_j. Similar quantities are defined for the eigenvalues $\bar{\zeta}_j$.

In ref. 1 it was shown how to express the first symplectic two-form in terms of the scattering data in the case $r = \pm\, q^*$. Tracing through the calculation there in the more general case, we find that

$$\begin{aligned} \Omega_1 &= i \int_{-\infty}^{\infty} \{ \delta q \wedge \delta r \}\, dx \\ &= \frac{i}{\pi} \int_{-\infty}^{\infty} \{ \delta \log b(\xi) \wedge \delta \log(\bar{a}(\xi)\, \bar{a}(\xi)) \}\, d\xi \; + \\ &\qquad - 2 \sum_{j=1}^{N} \{ \delta\, \zeta_j \wedge \delta \log b_j + \delta\, \bar{\zeta}_j \wedge \delta \log \bar{b}_j \}, \end{aligned} \tag{17}$$

where the last sum is absent if $r = +q^*$, since there are no bound states. When $r = \pm q^*$, then $\bar{a}(\xi) = a(\xi)^*$, and $\bar{b}(\xi) = \mp b(\xi)^*$. In this case one can choose canonically conjugate variables by letting

$$A_j = 4\eta_j, \qquad p_j = -4\xi_j, \qquad p(\xi) = -\frac{i}{\pi}\log|a(\xi)|,$$

represent the momenta (p's), and

$$B_j = \arg b_j, \qquad q_j = \log|b_j|, \qquad q(\xi) = \arg b(\xi).$$

the conjugate coordinates (q's) for the system. The first Hamiltonian functional is then expressed as

$$H_1 = \pm E = \frac{4}{\pi}\int_{-\infty}^{\infty}\xi^2\log(|a(\xi)|)\,d\xi - \frac{8i}{3}\sum_{j=1}^{N}(\zeta_j^3 - \bar{\zeta}_j^3). \tag{18}$$

From this expression, the quantized form follows directly as in ref. 1.

For the second symplectic form, we first recognize that by (12), (13) and ref. 7,

$$J_2 = L^A J_1 = L^A \sigma_2, \tag{19}$$

where L^A is the recursion operator for the squared eigenfunctions. Recall that the *squared eigenfunctions* corresponding to (15) are the functions

$$\Psi(\zeta,x) = \begin{pmatrix} v_1(\zeta,x)^2 \\ v_2(\zeta,x)^2 \end{pmatrix}.$$

We define the corresponding quantities Ψ_j for the bound states ζ_j similarly. The key result[10] is that the recursion operator L^A, given in (19), has the squared eigenfunctions as eigenstates:

$$L^A\Psi = \zeta\Psi, \qquad L^A\Psi_j = \zeta_j\Psi_j. \tag{20}$$

Thus we can compute the second symplectic form

$$\Omega_2 = \frac{1}{2}\langle \delta V^A | \wedge \sigma_2 (L^A)^{-1} | \delta V \rangle.$$

Now, according to (B3) of ref. 10,

$$\delta V = \frac{1}{\pi}\int_{-\infty}^{\infty} \{\ \delta\rho(\xi)\,\Psi(\xi) - \delta\bar{\rho}(\xi)\,\bar{\Psi}(\xi)\ \}\ d\xi\ -$$

$$-\,2\,i\ \sum_{j=1}^{N} (\ \delta\rho_j\,\Psi_j + \rho_j\,\delta\zeta_j\,\chi_j + \delta\bar{\rho}_j\,\bar{\Psi}_j + \bar{\rho}_j\,\delta\bar{\zeta}_j\,\bar{\chi}_j).$$

Therefore, using (20),

$$(L^A)^{-1}\,\delta V = \frac{1}{\pi}\int_{-\infty}^{\infty} \{\ \delta\rho(\xi)\,(L^A)^{-1}\,\Psi(\xi) - \delta\bar{\rho}(\xi)\,(L^A)^{-1}\,\bar{\Psi}(\xi)\ \}\ d\xi\ -$$

$$-\,2\,i\ \sum_{j=1}^{N} (\ \delta\rho_j\,(L^A)^{-1}\,\Psi_j + \rho_j\,\delta\zeta_j\,(L^A)^{-1}\,\chi_j + \delta\bar{\rho}_j\,(L^A)^{-1}\,\bar{\Psi}_j + \bar{\rho}_j\,\delta\bar{\zeta}_j\,(L^A)^{-1}\,\bar{\chi}_j).$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\delta\rho(\xi)\,\Psi(\xi)}{\xi + i\,\varepsilon} - \frac{\delta\,\bar{\rho}(\xi)\,\bar{\Psi}(\xi)}{\xi - i\,\varepsilon} \right) d\xi\ -$$

$$-\,2\,i\ \sum_{j=1}^{N} \left(\delta\left(\frac{\rho_j}{\zeta_j}\right)\Psi_j + \frac{\rho_j}{\zeta_j}\,\delta\zeta_j\,\chi_j + \delta\left(\frac{\bar{\rho}_j}{\bar{\zeta}_j}\right)\bar{\Psi}_j + \frac{\bar{\rho}_j}{\bar{\zeta}_j}\,\delta\bar{\zeta}_j\,\bar{\chi}_j \right),$$

where we have moved the integral over the continuous spectrum off the real axis to avoid the singularity at $\zeta = 0$. Therefore the only difference between the computation of Ω_1 and the new symplectic form Ω_2 are the weighting factors $1/\xi$ in the continuous spectrum, and $1/\zeta_j$ in the discrete spectrum. A similar calculation as was used to produce (17) now gives

$$\Omega_2 = \frac{i}{\pi}\int_{-\infty}^{\infty} \{\ \delta\log(\,\bar{a}(\xi)\,a(\xi)\,) \wedge \delta\arg b(\xi)\ \}\ \frac{d\xi}{\xi}\ +$$

$$+\ \frac{1}{2}\,\bar{b}(0)\,b(0)\,\delta\log\frac{\bar{a}(0)}{a(0)} \wedge \delta\log\frac{\bar{b}(0)}{b(0)}\ - \tag{21}$$

$$-\ 2\sum_{j=1}^{N} \left\{\ \delta\log\zeta_j \wedge \delta\log b_j + \delta\log\bar{\zeta}_j \wedge \delta\log\bar{b}_j\ \right\},$$

where the two complex integrals have combined to give the principal value in the leading term, and extra discrete term comes from the associated residues at the pole $\zeta = 0$. When $r = \pm\, q^*$, canonically conjugate variables are provided by the momenta

$$\hat{A}_j = 4 \arg \zeta_j, \qquad \hat{p}_j = -4 \log |\zeta_j|, \qquad \hat{p}(\xi) = -\frac{i}{\pi \xi} \log |a(\xi)|,$$

and the conjugate coordinates

$$\hat{B}_j = \arg b_j, \qquad \hat{q}_j = \log |b_j|, \qquad \hat{q}(\xi) = \arg b(\xi),$$

provided $\xi \neq 0$. In addition, the point $\xi = 0$ appears separately as the extra residue term in the expression for Ω_2, so this particular mode survives the principal value cancellation in a new discrete form. However, there is no simple formula for the relevant canonical variables there. Also, in the case $r = \pm q^*$, this term vanishes because $\bar{a}(0) = a(0)$, and so this extra complication does not arise. All the other modes for the continuous spectrum are related according to the simple reweighting

$$p(\xi) = \xi \, \hat{p}(\xi) . \tag{22}$$

For the second Hamiltonian structure, the Hamiltonian functional giving the nonlinear Schrödinger equation is the momentum (11). According to the calculations in ref. 1, it can be expressed in terms of the scattering data as

$$H_2 = P = \frac{4}{\pi} \int_{-\infty}^{\infty} \xi \log |a(\xi)| \, d\xi - 4i \sum_{j=1}^{N} (\bar{\zeta}_j^2 - \zeta_j^2). \tag{23}$$

Comparing with (18), we see that, in terms of the respective canonical variables, the continuous spectrum contribution is exactly the same weighted sum of the continuous canonical momentum variable associated with the respective symplectic two forms:

$$H_1 : \frac{4}{\pi} \int_{-\infty}^{\infty} \xi^2 p(\xi) \, dx \quad \text{versus} \quad H_2 : \frac{4}{\pi} \int_{-\infty}^{\infty} \xi \, p(\xi) \, dx = \frac{4}{\pi} \int_{-\infty}^{\infty} \xi^2 \hat{p}(\xi) \, dx.$$

Therefore, the continuous modes have identical quantizations. (The singular point $\xi = 0$ plays no role as both Hamiltonians make no contribution to this mode.) As for the bound states, we are reduced to the case of a collection of integrable two-dimensional Hamiltonian systems with different Hamiltonian structures. For the original symplectic form Ω_1, the Hamiltonian system corresponding to the discrete eigenvalue ζ_j has the form

$$(\log b_j)_t = -\frac{1}{2} \frac{\partial H_1}{\partial \zeta_j} = 4 i \zeta_j^2, \qquad (\zeta_j)_t = \frac{1}{2} \frac{\partial H_1}{\partial \log b_j} = 0,$$

and similarly for the eigenvalues $\bar{\zeta}_j$. (We are just reproducing the classical calculation of the evolution of the discrete scattering data for soliton equations.) For the second symplectic form Ω_2, the Hamiltonian system corresponding to the discrete eigenvalue ζ_j now takes the form

$$(\log b_j)_t = -\frac{1}{2}\frac{\partial H_1}{\partial \log \zeta_j} = 4\,i\,\zeta_j^2\,, \qquad (\log \zeta_j)_t = \frac{1}{2}\frac{\partial H_1}{\partial \log b_j} = 0\,,$$

and similarly for the eigenvalues $\bar{\zeta}_j$. Thus, these two dimensional Hamiltonian systems are identical, even though they use two different Hamiltonian structures:

$$-2\,\delta\,\zeta_j \wedge \delta \log b_j \qquad \text{versus} \qquad -2\,\delta \log \zeta_j \wedge \delta \log b_j\,.$$

However, as we remarked above, we take as fundamental the fact that a two-dimensional Hamiltonian system has a unique quantization, even though it has many different Hamiltonian structures. Therefore the bound states for the nonlinear Schrödinger equation also have identical quantizations. We conclude that both Hamiltonians lead to the same quantized version of the nonlinear Schrödinger equation.

As a final remark, we recall that the other soliton equations appearing in the AKNS scheme can be written in the form

$$\binom{q}{r}_t = \Omega(L^A)\binom{q}{r}_x\,,$$

where $\Omega(\xi)$ determines the linear dispersion relation[7]. These can all be written in biHamiltonian form using the same two Hamiltonian structures as above. An identical calculation, which we omit for the sake of brevity, will show that the two quantized versions of any member of these AKNS hierarchies will lead to the same quantum version. Moreover, it is not hard to see that the only difference between the quantized versions of two different members of the same soliton hierarchy is in the weighting factor $\Omega(\xi)$ for the modes corresponding to the continuous spectrum (with appropriate discrete contributions at the points where $\Omega(\xi) = 0$) and replacement of the discrete Hamiltonians by $\Omega(\zeta_j)$ and $\Omega(\bar{\zeta}_j)$ respectively. Thus the only distinction between the various quantized versions of a soliton hierarchy is in the weighting assigned to the continuous modes, and the replacement of the Hamiltonian governing the evolution of the bound states. Finally, we note that the same considerations will apply to other soliton equations, such as the Korteweg-deVries equation, as the key fact that the recursion operator is the squared eigenfunction operator remains valid.

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Time Evolution of the Scattering Data for the Forced Toda Lattice

By D. Wycoff and D. J. Kaup

Determination of the time evolution of the scattering data for an inverse scattering transform solution of the forced Toda lattice appears to require an overspecification of the boundary condition at the end of the lattice. This appears in the form of an apparent need to specify the values of *two* functions at the boundary rather than one. We present three different approaches to the resolution of this problem. One approach gives the Maclaurin series (in time) for the scattering data. The second approach gives the scattering data in terms of the solution to a nonlinear, nonlocal partial differential equation. The third approach gives the scattering data in terms of the solution to a linear integral equation. All three approaches reduce to *one* the number of functions which must be specified to determine a solution. The advantages and limitations of each approach are discussed.

1. Introduction

The inverse scattering transform (IST) [1] has proven to be a very powerful method, allowing us to fully solve the initial value problem for large classes of nonlinear partial differential equations and systems of ordinary differential equations. However, except for certain special cases [2–5], it has not been possible to solve initial–boundary value problems for these same integrable equations. The basic reason for this problem in treating the initial-BVP is easy to understand: the equations which determine the time evolution of the scattering

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data involve terms which *appear* to require an overdetermination of the problem [6, 7].[1]

The difficulties which arise are generic [6, 7], but let us consider a specific case, the forced Toda lattice [8–15] (FTL). By the FTL we mean a semiinfinite Toda lattice [16] in which the leftmost particle is externally forced to move in a specified way, $Q_0(t)$. No particular problems arise in setting up the IST for this problem [12, 17, 18]. Unfortunately, however, the differential equation which determines the time evolution of the scattering data for this IST [12] involves as a potential the motion, $Q_1(t)$, of the *second* particle in the lattice, which is not known until *after* the equations of motion have been solved. Although this situation is unpleasant, it is not surprising. In fact, if we solve the linear approximation to the FTL by Fourier transforms we find that the same thing occurs: the time evolution of the Fourier coefficients depends not only on $Q_0(t)$, which is given, but also on $Q_1(t)$. In the linear case this apparent need to overspecify the boundary condition is easily resolved, and $Q_1(t)$ can be found in terms of $Q_0(t)$. Since it has proven fruitful in the past to think of the IST as a form of Fourier analysis for nonlinear systems [1], it is worth considering whether this problem can be overcome in the nonlinear case as well.

Previous work on the forced Toda lattice has suggested that a careful consideration of the analytic properties of the scattering data might serve to resolve this apparent need to overspecify the boundary conditions [12, 14, 15]. However, these earlier results were useful only for very short times, since they involved calculating the terms in a Maclaurin series (in time) for the scattering data. In Section 3 we will show how to easily recover these Maclaurin series results.

In Sections 4 and 5 we establish an integral kernel representation for the scattering data and show how the analyticity requirement serves to restrict the kernels. We then use this restriction to find a single nonlinear, nonlocal partial differential equation, *involving no unknown potentials*, for the single independent kernel. In this formulation the only freedom is in the choice of the boundary condition, $Q_0(t)$. Thus, we have eliminated the apparent need to overspecify the boundary conditions in order to determine the time evolution of the scattering data. The appeal of this approach is that it seems to provide a direct, but nonlinear, pathway from specification of the boundary condition to calculation of the scattering data.

Of course, one of the major advantages of the IST approach has always been the fact that it gives the solution to nonlinear problems through linear means. This advantage is lost in the approach discussed above. In Section 7 we present an approach in which the kernel from which the scattering data is constructed is recovered from the solution to a *linear* integral equation. In writing down this linear integral equation we are free to choose only *one* function of one variable, $H_1(t)$. The scattering data for the forced Toda lattice is, of course, a function of *two* variables. Thus we have found a reduction of the problem from two variables to one. No other method has achieved such a reduction. The remaining unsolved

[1] Fokas [26] has recently established a nonlinear integral-differential equation (with singular kernel) for the time evolution of the scattering data of the nonlinear Schrödinger equation.

problem is to understand the mapping of the boundary condition, Q_0, into the function H_1.[2] Of course, that may be the most difficult part of the problem. If so, we have at least delineated what the remaining problem is, and have given the analytic structure of the solution.

2. Background

The forced Toda lattice [8–15] is a semiinfinite Toda lattice [16] with the leftmost particle driven by some external forcing. The equations of motion are

$$Q_n'' = \exp(Q_{n-1} - Q_n) - \exp(Q_n - Q_{n+1}), \qquad n = 1,2,3,\ldots. \tag{1}$$

(Here and in what follows time derivatives will be denoted by primes.) $Q_n(t)$ is the displacement of the nth particle in the lattice. $Q_0(t)$ is a specified boundary condition, and the initial data $\{Q_n(0), Q_n'(0)\}$ are given.

The scattering data for the forced Toda lattice can be recovered from a function $\chi(z,t)$ which satisfies the second order ordinary differential equation [12]

$$\chi'' + \left[\exp(Q_0 - Q_1) + \tfrac{1}{2}Q_0'' - \left(\tfrac{1}{2}Q_0' + \lambda\right)^2\right]\chi = 0. \tag{2}$$

λ is given in terms of the spectral parameter z by

$$\lambda = \frac{1}{2}\left(z + \frac{1}{z}\right). \tag{3}$$

Here $Q_0(t)$ is the position of the leftmost particle in the lattice, which is the externally specified boundary condition. $Q_1(t)$ is the position of the second particle in the lattice, which is of course unknown until *after* the problem is solved. Two different approaches have been used to try and resolve this difficulty. Numerical studies [13] have been used to make physically reasonable approximations to the function $Q_1(t)$. This approximation is then used in Equation (2) to find $\chi(z,t)$. The other approach, which we will use here, exploits the analytic properties of the inverse scattering transform for the forced Toda lattice [12, 14, 19, 20].

Analysis of the forced Toda lattice IST shows that the function $R(z,t)$ defined by

$$R(z,t) = \frac{1}{z}\chi(z,t)\exp\left[-\frac{1}{2}\left(z - \frac{1}{z}\right)t\right] \tag{4}$$

must be an analytic function of z inside the unit circle, $|z| = 1$ [12, 15]. In general,

[2] The inverse of this mapping is trivial. Given H_1, we can determine the boundary condition and the entire solution by solving our linear integral equation.

the solutions of Equation (2) will *not* have this property. Hansen and Kaup [14] were able to show that if the solutions $\chi(z,t)$ of Equation (2) are required to have this analytic property, then the Maclaurin series (in t) of the unknown potential $Q_1(t)$, and hence the Maclaurin series of the scattering data $\chi(z,t)$, are determined. This result is clearly of only limited practical use, since it only allows us to approximate the scattering data for very short times. In what follows we will show how to make progress toward a more useful result.

Let us rescale the quantities in Equation (2) in the following way:

$$\zeta = -2iz, \tag{5a}$$

$$S(\zeta,t) = \frac{1}{z}\chi(z,t), \tag{5b}$$

$$q(t) = -Q_0'(t), \tag{5c}$$

$$r(t) = \exp[Q_0(t)-Q_1(t)] - 1 + \tfrac{1}{2}Q_0''(t) - \tfrac{1}{4}[Q_0'(t)]^2. \tag{5d}$$

Equation (2) then becomes

$$S'' + [E^2 + ikq(t) + r(t)]S = 0, \tag{6}$$

where E and k are given by

$$E = \frac{\zeta}{4} + \frac{1}{\zeta}, \qquad k = \frac{\zeta}{4} - \frac{1}{\zeta}. \tag{7}$$

Finally, we note that the function $R(z,t)$ appearing in Equation (4) becomes

$$R(z,t) = S(\zeta,t)e^{-iEt}. \tag{8}$$

Therefore, $S(\zeta,t)e^{-iEt}$ must be an analytic function of ζ for $|\zeta| < 2$. The potential, $q(t)$ appearing in Equation (6) is given in terms of the *known* external forcing $Q_0(t)$ by Equation (5c). $r(t)$ is *unknown*, since it involves $Q_1(t)$, as shown in Equation (5d).

Equation (6) has been extensively studied for $t \in (-\infty,\infty)$ and with both potentials $q(t)$ and $r(t)$ given [21–24]. Here we will study Equation (6) for $t \in [0,\infty)$ with initial data

$$\begin{aligned} S(\zeta,0) &= 1, \\ S'(\zeta,0) &= iE + \tfrac{1}{2}q(0). \end{aligned} \tag{9}$$

The initial data (9) correspond to a lattice which is initially static, except for the zeroth particle, which is impulsed with velocity $-q(0)$.

3. Maclaurin series for the scattering data

Hansen and Kaup [14] have shown that the requirement that the scattering data have the correct analytic properties is sufficient to determine a Maclaurin series expansion (in time) for the scattering data. Let us see how this result follows very simply from the initial value problem (6), (9) and the analyticity requirement.

Define

$$F(\zeta,t) = S(\zeta,t)e^{-iEt-Q(t)}, \tag{10}$$

where

$$Q(t) = \frac{1}{2}\int_0^t q(s)\,ds. \tag{11}$$

F is an analytic function of ζ for $|\zeta|<2$, so we can write

$$F(\zeta,t) = \sum_{n=0}^{\infty} F_n(t)\left(\frac{i\zeta}{2}\right)^n, \qquad |\zeta| < 2. \tag{12}$$

Using this expansion in the initial value problem (6), (9) we find

$$F_0'(t) = 0, \tag{13a}$$

$$F_1'(t) = R'(t)F_0(t) + q(t)F_0'(t) + F_0''(t), \tag{13b}$$

$$F_n' = R'F_{n-1} + qF_{n-1}' + F_{n-1}'' + F_{n-2}' + qF_{n-2}, \qquad n \ge 2, \tag{13c}$$

with initial data

$$F_n(0) = \delta_{n,0}, \tag{14a}$$

$$F_n'(0) = 0, \tag{14b}$$

where $R(t)$ is given by

$$R(t) = \int_0^t \left(r(s) + \tfrac{1}{2}q'(s) + \tfrac{1}{4}[q(s)]^2\right) ds. \tag{15}$$

The differential equations (13) together with the initial data (14a) *alone* would determine $F_n(t)$ for $n = 0,1,2,3,\ldots$ if $R(t)$ were known. Since R contains r, it is in fact *unknown*, and the second initial datum (14b) serves to determine the Maclaurin series for $R(t)$.

Solving (13a) subject to the initial data (14a), we find

$$F_0(t) = 1. \tag{16}$$

The initial data (14b) are then automatically satisfied. Using (16) in (13b), together with the initial data (14a), we find

$$F_1(t) = R(t). \tag{17}$$

Requiring that (14b) also hold for $n = 1$ then gives

$$R'(0) = 0. \tag{18}$$

Similarly, solving (13c) for $n = 2$, subject to initial data (14a), gives

$$F_2(t) = \int_0^t [R'(s)R(s) + q(s)R'(s) + R''(s) + q(s)]\, ds \tag{19}$$

Requiring (14b) to hold for $n = 2$ then gives

$$R''(0) + q(0) = 0. \tag{20}$$

Continuing this process will generate all the coefficients $R^{(n)}(0)$ in the Maclaurin series for the unknown potential $R(t)$, and thus generate the Maclaurin series for the scattering data. To proceed beyond the Maclaurin series approach we must exhibit in a more explicit way the dependence of the scattering data on ζ.

4. Integral kernel representation for the scattering data

$S(\zeta, t)$ satisfying Equation (6) with initial data (9) can be expressed in terms of a pair of transformation kernels $N(t,s)$ and $L(t,s)$ as [15]

$$S(\zeta, t) = e^{Q(t)}\left\{e^{i\zeta t} + \int_{-t}^{t}\left[N(t,s) + \frac{i\zeta}{2}L(t,s)\right]e^{-i\zeta s}\, ds\right\}. \tag{21}$$

The kernels N and L satisfy a system of linear partial differential equations

$$\left[\partial_t^2 - \partial_s^2 + q(t)(\partial_t + \partial_s) + R'(t)\right]L(t,s) + q(t)N(t,s) = 0, \tag{22a}$$

$$\left[\partial_t^2 - \partial_s^2 + q(t)(\partial_t - \partial_s) + R'(t)\right]N(t,s) + q(t)L(t,s) = 0 \tag{22b}$$

with boundary conditions

$$N(t,t) = 0, \tag{23a}$$

$$L(t,t) = 0, \tag{23b}$$

$$N(t,-t) = -\tfrac{1}{2}R(t), \tag{23c}$$

$$L(t,-t) = -\tfrac{1}{2} + \tfrac{1}{2}e^{-2Q(t)}. \tag{23d}$$

If both $q(t)$ and $r(t)$ were given, the partial differential equations (22) with boundary conditions (23) would uniquely determine N and L. In our case $q(t)$ is given but $r(t)$, which appears in both the partial differential equations *and* the boundary conditions, is unknown. We will show below that by requiring $S(\zeta,t)$ to have the correct analytic properties we will obtain a relationship between the kernels N and L which serves to eliminate the unknown quantities.

5. Analyticity requirement

As discussed above, $S(\zeta,t)e^{-iEt}$ must be an analytic function of ζ inside the circle $|\zeta|=2$. In general, the solutions $S(\zeta,t)$ constructed from Equations (22) and (23) via equation (21) will *not* have this property. Therefore, we require

$$\oint_{|\zeta|=2} \zeta^n S(\zeta,t)e^{-iEt}\,d\zeta = 0, \qquad n = 0,1,2,\ldots. \tag{24}$$

Using the representation (21) of $S(\zeta,t)$ in Equation (24) and [25]

$$\oint_{|\zeta|=2} \left(\frac{i\zeta}{2}\right)^n e^{-iE\beta}\,d\zeta = 4\pi J_{n+1}(\beta), \tag{25}$$

where J_n is a Bessel function of the first kind, we find

$$\int_{-t}^{t} \left[N(t,s)J_n(t+s) + L(t,s)J_{n+1}(t+s)\right] ds = 0, \qquad n = 0,1,2,\ldots. \tag{26}$$

Thus N and L are *not* independent. Using [25]

$$J_{n+1}(u) = \int_0^u J_n(u-x)J_1(x)\frac{dx}{x}, \tag{27}$$

we find that if $S(\zeta,t)$ is to have the proper analytic structure in ζ, N must be given in terms of L by

$$N(t,s) = -\int_s^t L(t,u)\frac{J_1(s-u)}{s-u}\,du. \tag{28}$$

Equation (21) then gives the scattering data in terms of L alone:

$$S(\zeta,t) = e^{Q(t)}\left\{e^{iEt} + \int_{-t}^{t}\left[\kappa(t+s,E) + \frac{i\zeta}{2}\right]L(t,s)e^{-iEs}\,ds\right\}, \tag{29}$$

where

$$\kappa(z, E) = -\int_0^z \frac{J_1(x)}{x} e^{iEx}\, dx. \tag{30}$$

6. A partial differential equation for the scattering data involving no unknown potentials

We can use the relationship (28) between N and L in the partial differential equations (22) and the boundary conditions (23) to find a p.d.e. and boundary conditions for L alone which involve only the known potential $q(t)$.

First, notice that the boundary condition (23a) for N is automatically satisfied by (28). Using (28) in the boundary condition (23c), we find

$$R(t) = 2\int_{-t}^{t} du\, L(t,u) \frac{J_1(t+u)}{t+u}. \tag{31}$$

Using (28) and (31) in the p.d.e. (22a) we find an equation for L *alone*:

$$\left[\partial_t^2 - \partial_s^2 + q(t)(\partial_t + \partial_s) + R'(t)\right] L(t,s) - q(t)\int_s^t du\, L(t,u) \frac{J_1(s-u)}{s-u} = 0 \tag{32}$$

with the boundary conditions

$$\begin{aligned} L(t,t) &= 0, \\ L(t,-t) &= -\tfrac{1}{2} + \tfrac{1}{2} e^{-2Q(t)}. \end{aligned} \tag{33}$$

Equation (22b) is then satisfied identically, due to properties of the Bessel functions.

Neither the p.d.e. (32) nor the boundary conditions (33) involve any unknown potentials, since $R(t)$ is given in terms of L by Equation (31). Thus, given a function $q(t)$ (which represents the external forcing of the Toda lattice), $L(t,s)$ and hence the scattering data $S(\zeta, t)$ should be determined. The solutions to (32) would thus provide a mapping from boundary condition to scattering data. Unfortunately, the p.d.e. (32) is nonlinear *and* nonlocal, it is far from clear how to solve it even for a simple forcing, and the existence and uniqueness of solutions remains an open question. To try and avoid these problems we will next consider the linear integral (Gel'fand-Levitan-Marchenko) equations satisfied by N and L.

7. Linear integral equations for the kernels $N(t,s)$ and $L(t,s)$

$S(\zeta, t)$ satisfies the linear dispersion relation [15]

$$S(\zeta,t)e^{-iEt} = e^{-Q(t)} - \frac{1}{2\pi i}\int_A \frac{\rho(\zeta')\bar{S}(\zeta',t)e^{-iE't}}{\zeta'-\zeta}\,d\zeta' \tag{34}$$

where

$$\bar{S}(\zeta,t) \equiv S\left(-\frac{4}{\zeta},t\right). \tag{35}$$

The contour A passes above all the poles of the scattering data, $\rho(\zeta)$, in the complex ζ plane. (For details, see reference [15].) Using the representation (21) of $S(\zeta, t)$ in the linear dispersion relation (34), we find the following linear integral (GLM) equations for N and L:

$$N(t,z) - H_2(t-z) - \int_{-t}^{-z}[N(t,s)H_2(-s-z) + L(t,s)H_3(-s-z)]\,ds = 0, \tag{36a}$$

$$L(t,z) - H_1(t-z) - \int_{-t}^{-z}[N(t,s)H_1(-s-z) + L(t,s)H_2(-s-z)]\,ds = 0, \tag{36b}$$

where

$$H_n(z) \equiv \frac{1}{8\pi}\int_A \left(\frac{i\zeta}{2}\right)^{-n}\rho(\zeta)e^{-iEz}\,d\zeta. \tag{37}$$

The $\{H_n(z)\}$ are related by

$$H_{n+1} - H_{n-1} = 2H_n', \tag{38}$$

so *two* of them may be chosen independently, corresponding to the fact that in the original ordinary differential equation, (6), there are two independent potentials, $q(t)$ and $r(t)$. In what follows we will show that by requiring $S(\zeta, t)$ to have the correct analytic properties we reduce to *one* the number of functions $H_n(z)$ which can be independently specified.

Using the relationship (28) between N and L in the GLM equations (36), we find that the $\{H_n\}$ must be related by

$$H_2(z) = -\tilde{H}_1(z), \tag{39a}$$

$$H_3(z) = -\tilde{H}_2(z), \tag{39b}$$

where

$$\tilde{G}(z) \equiv \int_0^z G(z-x) J_1(x) \frac{dx}{x}. \tag{40}$$

Thus, once H_1 is given the other H_n are all determined. The relationship

$$H_3(z) - H_1(z) = 2H_2'(z) \tag{41}$$

required by Equation (38) then holds *identically*, due to properties of the Bessel functions. Using Equations (28) and (39a) in the GLM equation (36b), we find (after a change in the order of integration) a single linear integral equation for $L(t, z)$:

$$L(t,z) - H_1(t-z) + \int_{-t}^{t} \Gamma(t-z, s+z) L(t,s)\, ds = 0, \tag{42}$$

where the kernel Γ is given in terms of H_1 by

$$\Gamma(\alpha,\beta) \equiv \int_0^\alpha H_1(x) \frac{J_1(x+\beta)}{x+\beta}\, dx. \tag{43}$$

Thus, in Equation (42) we are free to specify only *one* function, $H_1(z)$.

This procedure for finding the time evolution of the scattering data $S(\zeta, t)$ of the forced Toda lattice can thus be summarized as follows:

1. Choose a single function $H_1(z)$.
2. Calculate $\Gamma(\alpha, \beta)$ from $H_1(z)$ using Equation (43).
3. Solve the linear integral equation (42) for $L(t, s)$.
4. Calculate which forcing $Q(t)$ led to this solution by using the boundary condition (23d) on $L(t, s)$.
5. Use Equation (29) to construct the scattering data $S(\zeta, t)$ from $L(t, s)$ and $Q(t)$.

Unfortunately, we do not know until after the fact *which* forcing led to our solution, since $Q(t)$ [and hence $q(t)$] is not known until *after* the linear integral equation (42) for $L(t, z)$ is solved. However, as shown in an appendix, if we are given a forcing $Q(t)$, we *can* calculate L iteratively from the linear integral equation (42) and the boundary condition (23d).

8. Summary and conclusions

We have outlined three approaches to the problem of finding the time evolution of the scattering data for the forced Toda lattice. All three approaches exploit the analytic properties of the Toda lattice inverse scattering transform.

The first approach, using Maclaurin series, has the obvious advantage of simplicity, since it deals directly with the ordinary differential equation satisfied by the scattering data. However, the results obtained by this method are of practical use only for very short times.

The second approach, as outlined in Section 6, gives us a single partial differential equation to solve, with the external forcing displayed explicitly. This indicates that specification of the boundary condition is sufficient to determine the scattering data for the forced Toda lattice. However, this p.d.e. is so unpleasant mathematically that it is hard to imagine that it gives any *practical* advantage over using direct numerical integration of the Toda lattice equations of motion.

The third approach, outlined in Section 7, involves the solution of a linear integral equation and so is probably the method which would be most useful in practice. However, it has the disadvantage that it appears to be impossible to determine *which* external forcing is being used until *after* the linear integral equation has been solved. This problem is avoided in the approach mentioned above, which deals directly with the partial differential equations (22). However, the approach outlined in Section 7 has the advantage of allowing the scattering data to be constructed through the solution of *linear* integral equations, while the approach using the partial differential equations is *nonlinear*. In our view, the outstanding problem is to determine the mapping from the boundary condition Q_0 to kernel H_1 which appears in the linear integral equation (42).

Appendix. Iterative solution of the linear integral equation for the scattering data

The linear integral equation (42) can be solved by iteration in powers of the external forcing $Q(t)$. In this appendix we present the results of such an expansion through second order. We expand L and its boundary conditions as

$$L(t,s) = \sum_{n=0}^{\infty} L^{(n)}(t,s),$$

$$L^{(n)}(t,t) = 0, \tag{A1}$$

$$L^{(n)}(t,-t) = \begin{cases} 0, & n=0, \\ \dfrac{1}{2}\dfrac{[-2Q(t)]^n}{n!}, & n=1,2,3,\ldots. \end{cases}$$

We also expand H_1 (and thus also Γ):

$$H_1(z) = \sum_{n=0}^{\infty} H_1^{(n)}(z). \tag{A2}$$

In zeroth order we find

$$
\begin{aligned}
H_1^{(0)}(z) &= 0,\\
L^{(0)}(t, z) &= 0.
\end{aligned}
\tag{A3}
$$

First order gives

$$
\begin{aligned}
H_1^{(1)}(t) &= -Q\left(\frac{t}{2}\right),\\
L^{(1)}(t, z) &= -Q\left(\frac{t-z}{2}\right).
\end{aligned}
\tag{A4}
$$

Finally, second order gives

$$
\begin{aligned}
H_1^{(2)}(t) &= \left[Q\left(\frac{t}{2}\right)\right]^2 + 4\int_0^{t/2} du \int_0^{t/2} dw\, Q(u)\frac{J_1(2w-2u)}{2w-2u}Q(w),\\
L^{(2)}(t, z) &= H_1^{(2)}(t-z) - 4\int_0^t du \int_0^{(t-z)/2} dw\, Q(u)\frac{J_1(2w-2u+t+z)}{2w-2u+t+z}Q(w).
\end{aligned}
\tag{A5}
$$

The iteration can clearly be continued to any order in the external forcing $Q(t)$.

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The Third-Order Singular-Perturbation Expansion of the Planar Cold-Fluid Magnetron Equations

By D. J. Kaup and Gary E. Thomas

The singular-perturbation expansion of the plasma cold-fluid equations for crossed fields in a planar geometry is presented. The general expansion is carried out to third order. Various instabilities that occur in the first, second, and third orders are discussed.

I. Introduction

Magnetrons and other crossed-field devices have been in existence since the 1940s [1]. Over the decades a broad technology base for building and designing such devices [2] has been developed. However, most of this technology is a result of employing empirical methods. The major reason for this is that the strongly nonlinear nature of the basic crossed-field interaction mechanism has made it very difficult to obtain a good understanding of the physical processes which occur in these devices. Without this understanding, accurately predicting the operation of crossed-field devices has not been possible.

Motivated by the need to predict and improve the basic performance of these devices for radar and other applications, a novel approach to developing a strongly nonlinear theory was undertaken. This approach was to apply soliton techniques to modeling the nonlinear interaction mechanism in crossed-field devices [3–5]. Given that it would be extremely difficult to solve the fully nonlinear problem, a model problem was chosen which could contain enough

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truth to allow some predictions to be made that could be checked against available experimental data. The early work [3–5] showed close agreement between predictions of the strongly nonlinear theory and available data, implying that the basic approach was indeed valid. As a consequence, extensive effort has been undertaken to place this novel approach on a rigorous footing by working to bridge the gap between linear theory and this new, strongly nonlinear theory through the use of standard multiscale perturbation methods. In addition, it was hoped that assumptions used in the original work could either be further justified or removed entirely.

In this paper the geometry used for the multiscale perturbation expansion is the planar magnetron shown in Figure 1. This device has a cathode at $y = 0$ and a parallel anode at $y = l$, with a large positive voltage difference V applied across the anode-cathode interelectrode region. In addition, one has a perpendicular magnetic field which is sufficiently large to return any emitted electrons to the cathode, thereby producing "magnetic insulation." We shall ignore any z-dependence, and assume that the device is infinite in extent in the z-direction and that the solution is invariant under any z-translations.

In actual devices, there is also either a "slow-wave structure" or a set of vanes on the anode. The purpose of these structures is simply to select and/or limit the possible wavenumbers which will propagate. For simplicity, we shall assume the anode to be smooth, realizing that any solution we find could be either limited or enhanced depending on these structures.

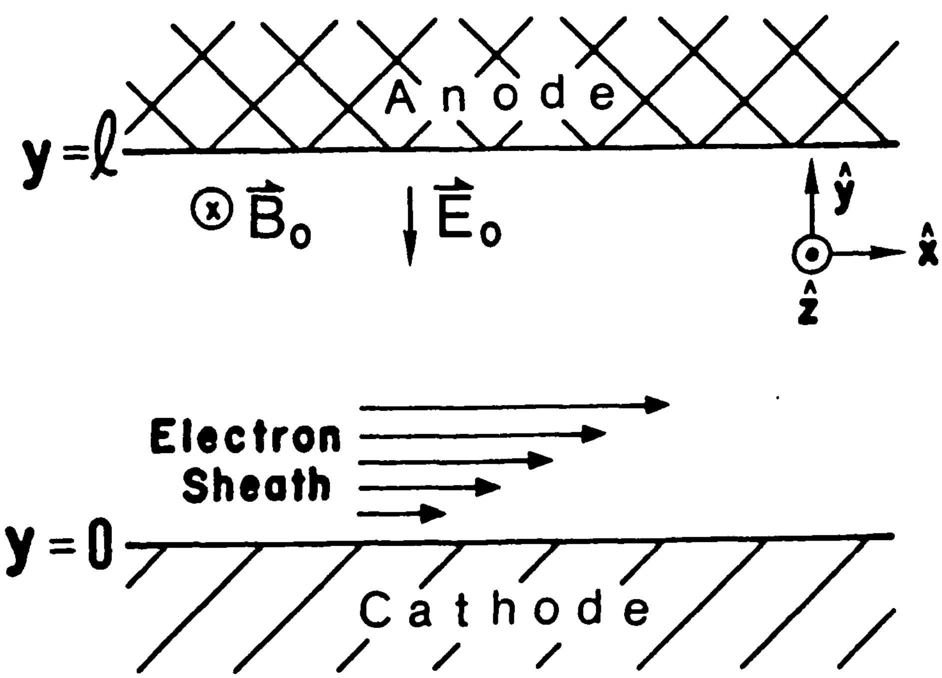

Figure 1. Geometry and the shear flow in the planar magnetron.

At startup, one first turns on the magnetic field, then the voltage. What is thought to occur is a rapid buildup of an electron sheath around the cathode. The final sheath is considered to be a laminar shear flow of electrons caused by the $\mathbf{E}\times\mathbf{B}$ drift. One expects the number density of the sheath to increase until the electric field vanishes at the cathode due to the space charge above the cathode. When this occurs, then no more current can be drawn from the cathode, and presumably an equilibrium is established. However, at first there is nothing which specifies or determines the density profile of this sheath. It is usually assumed that the density profile is box-shaped, as shown in Figure 2(a), and that the density is a maximum ($\omega_p^2 = \Omega_c^2$). However, there is nothing in the zeroth-order or the first-order cold-fluid equations to prevent other density profiles from occurring, except that density profiles with positive density gradients are known to be strongly unstable [6–8] and tend to relax to profiles without positive density gradients. Thus the combination box-and-ramp profile shown in Figure 2(b) is also a valid profile to consider.

After the sheath is formed, what happens next is not entirely clear. The classical explanation is that a high-wavenumber, linear instability exists [9]. Due to thermal noise, this instability will be excited, will grow, and eventually will saturate. While there seems to be nothing wrong with this explanation, nevertheless nobody has yet built a device based on it. Neither does this explanation explain the narrow operating range of such devices. This indicates that either the theory is wrong, or essential physics has been omitted.

In this paper, we describe the singular perturbation expansion of the cold-fluid equations for the planar magnetron. Each order in the expansion (out to third order), is reviewed: we describe the main features of that order, how it could interrelate with the other orders, and what its consequences are, and we list any important open questions. The purpose of this is simply to detail what is certain, to clarify what is unknown, and to attempt to specify what must be determined.

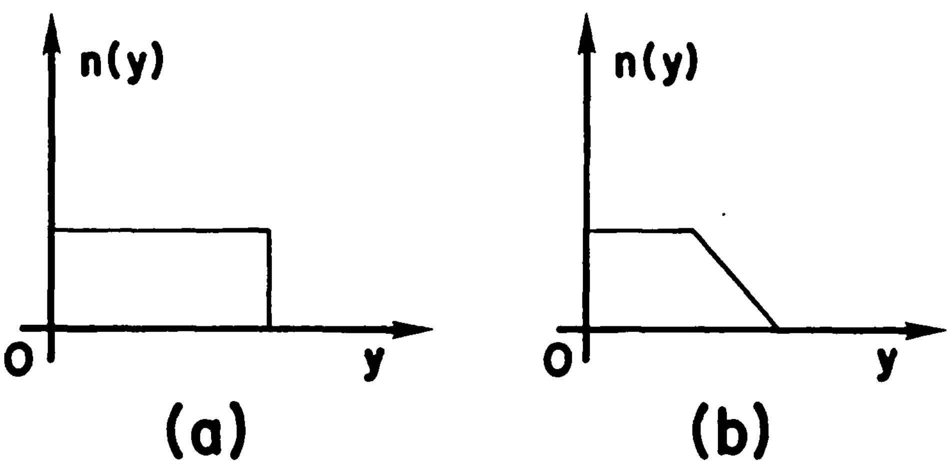

Figure 2. Two standard density profiles: (a) Brillouin flow (square profile) and (b) square-plus-ramp profile.

We start with the cold-fluid, nonneutral, single-species, plasma equations coupled to Poisson's equation:

$$\partial_t n + \nabla\cdot(n\mathbf{v}) = 0, \tag{1a}$$

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla\phi + \mathbf{v}\times\mathbf{\Omega} = 0, \tag{1b}$$

$$\nabla^2\phi - n = 0, \tag{1c}$$

where ϕ is (e/m) times the electrostatic potential, n is the plasma frequency squared, $\mathbf{v}$ is the velocity, and $\mathbf{\Omega} = -\Omega\hat{\mathbf{k}}$, where Ω is the electron cyclotron frequency. In lowest order, the system shall be independent of x and t, with only a y-dependence. On top of this, we shall impose a small rf plane wave, which will drive the entire system, and we shall follow the consequences of this perturbation out to third order. In addition to the plane wave, we shall also assume a long-wavelength modulation of the plane wave, parallel to the anode. Thus we expand all our physical quantities $(n, \phi, \mathbf{v})$ as

$$n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \varepsilon^3 n_3 + \cdots, \tag{2}$$

where ε indicates the amplitude of the perturbing rf wave. We expand the high-order terms as

$$n_1 = \left[\hat{n}_1 e^{i(kx-\omega t)} + \text{c.c.}\right] + n_1^{(0)}, \tag{3a}$$

$$n_2 = \left[\hat{n}_2^{(2)} e^{2i(kx-\omega t)} + \text{c.c.}\right] + \left[\hat{n}_2 e^{i(kx-\omega t)} + \text{c.c.}\right] + n_2^{(0)} \tag{3b}$$

where k is the parallel wavenumber and Ω is the frequency. All the above coefficients $[n_0, \hat{n}_1, n_1^{(0)}, \hat{n}_2^{(2)}, \hat{n}_2, n_2^{(0)}$, etc.] are considered to be functions of the following slow variables:

$$\chi = \varepsilon x, \tag{4a}$$

$$T = \varepsilon t, \tag{4b}$$

$$\tau = \varepsilon^2 t, \tag{4c}$$

$$\tau' = \varepsilon^3 t. \tag{4d}$$

Needless to say, unless otherwise stated, all quantities are also functions of y, which is assumed to be of the same scale length as x.

II. Zeroth order

This order specifies the background on which the rf wave will propagate. The resulting equations are

$$\mathbf{v}_0 = v_0\hat{\mathbf{i}}, \qquad v_0 = \partial_y\phi_0/\Omega, \tag{5a}$$

$$\partial_y^2\phi_0 = n_0, \tag{5b}$$

where $n_0(y)$ is arbitrary. Imposing the space-charge-limited current condition gives

$$\partial_y\phi_0(y=0) = 0, \tag{5c}$$

which, with $\phi_0(y=0)=0$ and the specification of $n_0(y)$, uniquely determines the solution. Of course, $\phi_0(y=0)$ must be the applied anode-cathode voltage, which places one restriction on the choice of $n_0(y)$. In general, n_0 may also be dependent on the slow variables χ, T, τ, and τ'. But these dependencies will not enter until the higher orders.

III. First-order DC

This order describes how the background responds to the long spatial scale. Since $n_0(y)$ is arbitrary except for the voltage condition, we may scale $n_1^{(0)}$ to zero by redefining $n_0+\epsilon n_1^{(0)}$ to be the new n_0, without any loss of generality. Then we have

$$\partial_y\left(n_0 v_{1y}^{(0)}\right) = S_{1n}^{(0)}, \tag{6a}$$

$$-\frac{\Delta^2}{\Omega}v_{1y}^{(0)} = S_{1x}^{(0)}, \tag{6b}$$

$$\Omega v_{1x}^{(0)} - \partial_y\phi_1^{(0)} = S_{1y}^{(0)}, \tag{6c}$$

$$\partial_y^2\phi_1^{(0)} = S_{1\phi}^{(0)}, \tag{6d}$$

where the sources are

$$S_{1n}^{(0)} = -\partial_T n_0 - \partial_\chi(n_0 v_0), \tag{7a}$$

$$S_{1x}^{(0)} = -\partial_T v_0 - v_0\partial_\chi v_0, \tag{7b}$$

$$S_{1y}^{(0)} = 0, \tag{7c}$$

$$S_{1\phi}^{(0)} = 0, \tag{7d}$$

and we have introduced

$$\Delta^2 = \Omega^2 - n_0, \tag{8}$$

which is assumed to be positive definite.

From Poisson's equation (6d) and (7d), if the voltage difference between the anode and cathode is to remain fixed, then the only solution is

$$\phi_1^{(0)} = 0, \tag{9a}$$

which with (6c) and (7c) gives

$$v_{1x}^{(0)} = 0. \tag{9b}$$

Now (6b) gives the solution of

$$v_{1y}^{(0)} = \frac{\Omega}{\Delta^2}\left(\partial_T v_0 + v_0 \partial_\chi v_0\right), \tag{9c}$$

while (6a) and (7a) may be integrated and combined with (9c), which results in

$$\Omega \partial_T \partial_y \phi_0 + \left(\partial_y \phi_0\right) \partial_y \partial_\chi \phi_0 - n_0 \partial_\chi \phi_0 - \Delta^2 \Omega C_1 = 0, \tag{10}$$

where $C_1(\chi, T, \tau)$ is a constant of integration. This constant can be evaluated for the fixed voltage condition by simply integrating (10) from the anode to the cathode. Since

$$\partial_T \phi_0(y=0) = 0 = \partial_T \phi_0(y=l), \tag{11}$$

one finds

$$\Omega C_1 = \frac{\partial_\chi U}{\int_0^l \Delta^2 dy}, \tag{12a}$$

where

$$U = \int_0^l \left(\partial_y \phi_0\right)^2 dy \tag{12b}$$

is proportional to the electrostatic energy stored between the cathode and anode. An interpretation of C_1 follows upon differentiating (10). This gives

$$\partial_T n_0 + \left(C_1 - \frac{\partial_\chi \phi_0}{\Omega}\right) \partial_y n_0 + \frac{\partial_y \phi_0}{\Omega} \partial_\chi n_0 = 0. \tag{13}$$

Obviously, (13) shows that lines of constant n_0 are convected by the flow

$$\mathbf{u} = \hat{\mathbf{i}}\frac{\partial_y \phi_0}{\Omega} + \hat{\mathbf{j}}\left(C_1 - \frac{\partial_x \phi_0}{\Omega}\right) \tag{14}$$

Thus C_1 is a vertical velocity, in addition to the $\mathbf{E}\times\mathbf{B}$ drift velocity, which convects n_0, and arises from the fixed anode-cathode voltage. In contrast to this, if we had used the fixed charge conditions on the anode and cathode, then we would have required $\partial_y\phi_0(y=l)$ to be fixed. Assuming $n_0(y=l)=0$, Equation (10) would then give $C_1=0$, upon evaluating it at $y=l$. In the fixed-voltage case, the direction in which lines of constant n_0 are convected could be either up or down, depending on the values of C_1 and $\partial_x\phi_0$. In general, if $\partial_x U>0$, the convection tends to be more upward, while if nearby the value of $\partial_x U$ is negative, then in that region the convection tends to be more downward. As a consequence of this, we would expect a vortex cell to form wherever this counterflowing convection occurred.

Although the evolution equation (10) is fully nonlinear, it still could be analyzed in the weakly nonlinear limit by additional singular-perturbation expansion, which would result in the KdV equation [10–12]. A report on this is currently being prepared. An important consequence of this is that once $\partial_x\phi_0$ becomes nonzero and it is positive somewhere and negative somewhere else, then eventually at least one KdV soliton will grow and form.

At startup, one would like to consider the background to be smooth and stationary so that the above equation could be ignored, at least until some linear or nonlinear instabilities have had time to grow up out of any noise. However, once any long-scale variations have been introduced into the background, then this order can no longer be ignored, and it will require the fully nonlinear evolution equation (10) to describe the evolution of these long-scale background variations.

IV. First-order fundamental

This section covers the same material as in the short-wavelength linear stability theory. The classic work of Buneman, Levy, and Lindson [9] is the usual reference on this order. However, they only considered the box profile in Figure 2(a), as have many others since [13–15]. More recently, Davidson and others [6–8] have considered various other profile shapes in studies of the resonant diocotron instability. However he has mostly emphasized profiles with positive density gradients, since these demonstrate the strongest resonant diocotron instability.

As far as is known to us, no linear stability analysis has ever been done on the ramped profile shown in Figure 2. To understand why this latter profile would be of importance we point out that in the absence of a finite nonzero density gradient, certain terms vanish from the cold-fluid equations. These terms are also the terms which are responsible for the resonant diocotron instability. Thus the box profile can have different modes than the ramped profile has.

For now, let us present the linear equations of this order. They are

$$-i\omega_e\hat{n}_1 + ikn_0\hat{v}_{1x} + \partial_y\left(n_0\hat{v}_{1y}\right) = 0, \tag{15a}$$

$$-i\omega_e\hat{v}_{1x} - ik\hat{\phi}_1 - \frac{\Delta^2}{\Omega}\hat{v}_{1y} = 0, \tag{15b}$$

$$-i\omega_e\hat{v}_{1y} - \partial_y\hat{\phi}_1 + \Omega\hat{v}_{1x} = 0, \tag{15c}$$

$$\left(\partial_y^2 - k^2\right)\hat{\phi}_1 - \hat{n}_1 = 0 \tag{15d}$$

where we have introduced

$$\omega_e = \omega - kv_0, \tag{16}$$

which is the frequency as seen by the streaming electrons. Since the anode and cathode are conductors, we must impose the boundary conditions

$$\hat{\phi}_1(y=0) = 0 = \hat{\phi}_1(y=l). \tag{17}$$

In general, this overdetermines the solution except for certain values (complex in general) of ω, which are the discrete eigenfrequencies, and the corresponding solution is called the eigenmode. These eigenvalues depend on the density profile not only through n_0 in (15a), but also through v_0 in (16) and Δ^2 in (15b). The standard way to solve for the eigenspectrum has been to reduce (15) to a second-order differential equation for $\hat{\phi}_1$, as in Reference [7]. However, this introduces a false singularity wherever $\omega_e^2 = \Omega^2$. This can be avoided by reducing (15) to a second-order differential system for the velocities instead. Thus let us take

$$\hat{v}_{1x} = iu\psi/k, \tag{18a}$$

$$\hat{v}_{1y} = p\psi/A, \tag{18b}$$

$$\hat{\phi}_1 = i\psi\left(\frac{p\Delta^2}{Ak\Omega} - \frac{u\omega_e}{k^2}\right), \tag{18c}$$

$$\hat{n}_1 = ip\psi Q, \tag{18d}$$

where p and u are functions of y, ψ is an amplitude and is only a function of the slow variables, and

$$Q = \frac{2kn_0^2}{\Omega A^2} - \frac{\partial_y n_0}{A\omega_e}, \tag{19a}$$

$$A = \Omega^2 - \omega_e^2, \tag{19b}$$

If we also define

$$B = \frac{k^2}{A} - \frac{k\partial_y n_0}{\Omega\omega_e A} - \frac{2k^2 n_0 \Delta^2}{\Omega^2 A^2}, \tag{19c}$$

then (15) reduces very nicely to the linear second-order system

$$\partial_y p = Au, \tag{20a}$$

$$\partial_y u = Bp, \tag{20b}$$

which only has singularities at $\omega_e = 0$ and $\pm\Omega$. From (17) and (18c), one can obtain the boundary values for p and u.

V. Second-order DC

This contains the quasilinear theory of Davidson [7] and also has been covered in [16]. The equations are form-wise identical to the first-order DC equations, (6), except for the sources. They are

$$\partial_y\left(n_0 v_{2y}^{(0)}\right) = S_{2n}^{(0)}, \tag{21a}$$

$$-\frac{\Delta^2}{\Omega} v_{2y}^{(0)} = S_{2x}^{(0)}, \tag{21b}$$

$$\Omega v_{2x}^{(0)} - \partial_y \phi_2^{(0)} = S_{2y}^{(0)}, \tag{21c}$$

$$\partial_y^2 \phi_2^{(0)} = S_{2\phi}^{(0)}, \tag{21d}$$

where as before, we choose $n_2^{(0)} = 0$ without loss of generality. Now, the sources are

$$S_{2n}^{(0)} = -\partial_\tau n_0 - \partial_y\left(\hat{n}_1^* \hat{v}_{1y} + \text{c.c.}\right), \tag{22a}$$

$$S_{2x}^{(0)} = -\partial_\tau v_0 - \left(\hat{v}_{1y}^* \partial_y \hat{v}_{1x} + \text{c.c.}\right), \tag{22b}$$

$$S_{2y}^{(0)} = -\partial_T v_{1y}^{(0)} - v_0 \partial_X v_{1y}^{(0)} - \partial_y\left(\hat{v}_{1y}^* \hat{v}_{1y}\right) - \left(ik\hat{v}_{1x}^* \hat{v}_{1y} + \text{c.c.}\right), \tag{22c}$$

$$S_{2\phi}^{(0)} = -\partial_X^2 \phi_0, \tag{22d}$$

where c.c. stands for the complex conjugate of the preceding term.

Solution is now straightforward. Equations (21d) and (22d) yield a unique solution for $\phi_2^{(0)}$ upon imposing the boundary conditions

$$\phi_2^{(0)}(y=0) = 0 = \phi_2^{(0)}(y=l). \tag{23}$$

Then (21c) and (22c) yield the solution for $v_{2x}^{(0)}$, (21b) and (22b) yield $v_{2y}^{(0)}$, and (21a) and (22a), upon integration, give the condition

$$\partial_\tau \partial_y \phi_0 - \Delta^2 C_2 = -\frac{\Delta^2}{\Omega^2}(\hat{n}_1^* \hat{v}_{1y} + \text{c.c.}) - \frac{n_0}{\Omega}(\hat{v}_{1y}^* \partial_y \hat{v}_{1x} + \text{c.c.}),$$

where C_2 is the constant of integration. Making use of the solution (18), this becomes

$$\partial_\tau \partial_y \phi_0 - \Delta^2 C_2 = D \partial_y n_0, \tag{24}$$

where

$$D = \sum_k D_k \tag{25a}$$

and

$$D_k = \frac{2\gamma(\psi^* \psi)(p^* p)}{(\omega_e^* \omega_e)(A^* A)} e^{2\gamma t}, \tag{25b}$$

and γ is the imaginary part of ω. (If we had more than one linear mode present, then the total D would be equal to a sum over all modes [16] of the individual D_k's.) Differentiating (24) once more gives

$$\partial_\tau n_0 + C_2 \partial_y n_0 = \partial_y(D \partial_y n_0), \tag{26}$$

which one immediately recognizes as a diffusion equation for the density. As before, the constant C_2 is determined by integrating (24) from the anode to the cathode under fixed-voltage conditions. This give

$$C_2 = \frac{\int_0^l D(-\partial_y n_0)\, dy}{\int_0^l \Delta^2\, dy}, \tag{27}$$

which for $\partial_y n_0 < 0$ and $D > 0$ gives $C_2 > 0$.

There are several consequences of (26). First, although a general arbitrary initial condition would have a combination of modes with both positive and negative values of γ, due to the $e^{2\gamma t}$ term in (25) one expects the unstable modes to eventually dominate, thereby driving the total D positive.

Second, as a function of y, D is largest wherever the magnitude of ω_e or A is smallest. The values of y where this occurs are the positions of resonances. We define the diocotron resonance to be where the real part of ω_e vanishes, and the cyclotron resonances to be where the real part of ω_e is equal to either $+\Omega$ or $-\Omega$. Thus by (25b), diffusion of the density is maximum at these resonances. This maximum rate of diffusion will cause plateaus to form wherever an unstable mode has any resonances [16].

Third, since the diffusion rate is proportional to the density gradient, a box profile, which has an infinite discontinuity, is unstable against diffusion, and will instantaneously shift toward the ramped profile shown in Figure 2(b).

Fourth, the positive value of C_2 implies that there should be a net upward flux of particles, since the operator $\partial_\tau + C_2\partial_y$ will tend to move lines of constant n_0 upwards.

Fifth, this diffusion only occurs for unstable modes. If all modes are stable, then n_0 does not diffuse or even change, and the background remains undisturbed.

Sixth, any slow spatial scales in ψ (such as the scale χ) will be forced into the background if $D \neq 0$. This follows from (24) and (25). If the linear amplitudes ψ are functions of χ, then so is D, and then within the time scale of τ, the same slow spatial scales will appear in n_0, due to diffusion. When this occurs, the first-order DC equation, (10), becomes excited, and the density then starts changing on the faster time scale T.

Seventh, (24) allows equilibrium profiles to exist. As mentioned before, in zeroth and first order, there are no conditions which act to determine the profile shape of $n_0(y)$. However, here in second order, due to the diffusion mechanism, there can exist profiles where n_0 will be static. To see this, setting $\partial_\tau\phi_0 = 0$ in (24) gives

$$\Delta^2 = \Delta^2(0)\exp\left(C_2\int_0^y \frac{dy}{D(y)}\right). \tag{28}$$

Note that if there is one dominant unstable mode, then by (27), the ratio of D/C_2 is time-independent; then (28) gives a unique profile shape, which presumably the solution of (26) would relax toward. However, determining this equilibrium profile requires knowing D, which requires the eigenfunction $p(y)$ [see (25b)]. But the eigenfunctions depend on the background profile, so, we find ourselves in a circular situation. How one can break out of this is not known. Although numerically solving the coupled system of equations (20) and (24) should converge to some equilibrium profile, still that problem remains to be treated.

Lastly, I wish to point out that quasilinear diffusion is not the only important consequence of this order. Even if there are no unstable modes ($D = 0$) and n_0 is stationary, then there is *always* at least a shift in the shear flow, even if the background has no slow spatial scales. From (18), (20), (21), and (22), one finds

$$v_{2x}^{(0)} = \frac{1}{\Omega}\partial_y\phi_2^{(0)} - (\partial_T + v_0\partial_\chi)\left(C_1 + \frac{n_0\partial_\chi\phi_0}{\Omega\Delta^2}\right) - \frac{\psi^*\psi}{\Omega p^*p}\partial_y\left(\frac{(p^*p)^2}{A^*A}\right)e^{2\gamma t}, \tag{29a}$$

$$v_{2y}^{(0)} = \frac{1}{\Delta^2}\partial_\tau\partial_y\phi_0 + \psi^*\psi\frac{p^*p}{kA^*A}(iAB + \text{c.c.})e^{2\gamma t}. \tag{29b}$$

[If one had several linear modes present, then the last terms in both (29a) and (29b) would be replaced by the appropriate sums over k.] These last terms in (29) are always present regardless whether all modes are stable or not.

VI. Second-order fundamental

The main consequence of this order is to determine the group velocity. Since it may be useful at times to evaluate these higher order solutions numerically, we shall briefly discuss several methods which would be useful. The equations of this order are formwise identical to the first-order fundamental equations (15), except that in this order they have sources. They are

$$-i\omega_e\hat{n}_2 + ikn_0\hat{v}_{2x} + \partial_y(n_0\hat{v}_{2y}) = S_{2n}^{(1)}, \tag{30a}$$

$$-i\omega_e\hat{v}_{2x} - ik\hat{\phi}_2 - \frac{\Delta^2}{\Omega}\hat{v}_{2y} = S_{2x}^{(1)}, \tag{30b}$$

$$-i\omega_e\hat{v}_{2y} - \partial_y\hat{\phi}_2 + \Omega\hat{v}_{2x} = S_{2y}^{(1)}, \tag{30c}$$

$$\partial_y^2\hat{\phi}_2 - k^2\hat{\phi}_2 - \hat{n}_2 = S_{2\phi}^{(1)}, \tag{30d}$$

where

$$S_{2n}^{(1)} = -\partial_T\hat{n}_1 - \partial_\chi(v_0\hat{n}_1) - \partial_\chi(n_0\hat{v}_{1x}) - \partial_y(\hat{n}_1 v_{1y}^{(0)}). \tag{31a}$$

$$S_{2x}^{(1)} = -\partial_T\hat{v}_{1x} - \partial_\chi(v_0\hat{v}_{1x}) + \partial_\chi\hat{\phi}_1 - v_{1y}^{(0)}\partial_y\hat{v}_{1x}, \tag{31b}$$

$$S_{2y}^{(1)} = -\partial_T\hat{v}_{1y} - v_0\partial_\chi\hat{v}_{1y} - \partial_y(v_{1y}^{(0)}\hat{v}_{1y}), \tag{31c}$$

$$S_{2\phi}^{(1)} = -2ik\partial_\chi\hat{\phi}_1. \tag{31d}$$

If we take the background to be independent of the slow variables, then these sources are linear in $\partial_T\psi$ and $\partial_\chi\psi$. Thus the general solution will be of the form

$$\hat{v}_{2x} = i\frac{u_{2T}\partial_T\psi + u_{2\chi}\partial_\chi\psi}{k}, \tag{32a}$$

$$\hat{v}_{2y} = \frac{p_{2T}\partial_T\psi + p_{2\chi}\partial_\chi\psi}{A}, \tag{32b}$$

$$\hat{\phi}_2 = i(\partial_T\psi)\left(\frac{p_{2T}\Delta^2}{Ak\Omega} - \frac{u_{2T}\omega_e}{k^2}\right) + i(\partial_\chi\psi)\left(\frac{p_{2\chi}\Delta^2}{Ak\Omega} - \frac{u_{2\chi}\omega_e}{k^2}\right)$$
$$+ u\frac{\partial_T\psi + v_0\partial_\chi\psi}{k^2} + (\partial_\chi\psi)\frac{u\omega_e/k - p\Delta^2/(A\Omega)}{k^2}, \tag{32c}$$

$$\hat{n}_2 = i(\partial_T\psi)p_{2T}Q + i(\partial_\chi\psi)p_{2\chi}Q + 2pn_0^2\frac{\partial_\chi\psi}{\Omega A^2}$$
$$- p(\partial_T\psi + v_0\partial_\chi\psi)\left(\frac{\partial_y n_0}{A\omega_e^2} + \frac{4k\omega_e n_0^2}{A^3\Omega}\right). \tag{32d}$$

where p_{2T}, $p_{2\chi}$, u_{2T}, and $u_{2\chi}$ satisfy the equations ($\alpha = 2T$ or 2χ)

$$\partial_y p_\alpha - A u_\alpha = R_\alpha, \tag{33a}$$

$$\partial_y u_\alpha - B p_\alpha = \tilde{R}_\alpha \tag{33b}$$

with

$$R_{2T} = -2ikn_0 p \frac{\Omega^2 + \omega_c^2}{\Omega A^2}, \tag{34a}$$

$$R_{2\chi} = v_0 R_{2T} + i\frac{2pn_0\omega_c}{A\Omega} - i\frac{uA}{k}, \tag{34b}$$

$$\tilde{R}_{2T} = ip\left(k\frac{\partial_y n_0}{\Omega\omega_c^2 A} - \frac{4k^2 n_0 \Delta^2 \omega_c}{A^3\Omega^2}\right), \tag{34c}$$

$$\tilde{R}_{2\chi} = v_0 \tilde{R}_{2T} - ipk\frac{A\Omega^2 - 2n_0\Delta^2}{A^2\Omega^2}. \tag{34d}$$

Let us now describe how the solution in this order could be constructed. The boundary conditions in this order are again

$$\hat{\phi}_2(y=0) = 0 = \hat{\phi}_2(y=l). \tag{35}$$

To the solutions for (p_α, u_α) we can always add an arbitrary amount of the homeogeneous solution (p, u). This has the consequence that we may arbitrarily set either p_α or u_α equal to zero at $y=0$. At $y=0$, it is then possible to set the coefficients of both $\partial_T\psi$ and $\partial_\chi\psi$ in $\hat{\phi}_2$, (32c), equal to zero, by judiciously choosing the initial values for p_α and u_α. One then integrates (33) forward to the anode. The solutions for p_α and u_α are now unique. In general, $\hat{\phi}_2$ evaluated at the anode will be of the form $f_T\partial_T\psi + f_\chi\partial_\chi\psi$, where f_T and f_χ are two constants. Thus (35) then demands

$$\partial_T\psi + c\partial_\chi\psi = 0, \tag{36}$$

where $c = f_\chi/f_T$ is the group velocity. The major consequence of this order is that the rf amplitude ψ will evolve with the group velocity c on the time scale of $T = \epsilon t$. The value of c may be determined by the above procedure of integrating (33).

An alternative procedure is to use the Wronskian relationship. From (32a), (32b), (33) and (20), we have

$$\partial_y\left(Au\hat{v}_{2y} + ikp\hat{v}_{2x}\right) = \left(\partial_T\psi\right)\left(uR_{2T} - p\tilde{R}_{2T}\right) + \left(\partial_\chi\psi\right)\left(uR_{2\chi} - p\tilde{R}_{2\chi}\right) \tag{37}$$

Integrating (37) from the anode to the cathode results in (36) and the initial and final values of $\hat{v}_{2x}$ and $\hat{v}_{2y}$, which follow from the boundary conditions (35).

A last alternative method for calculating c is directly from

$$c = \frac{d\omega}{dk}. \tag{38}$$

If one differentiates (15) with respect to k and χ, identifies

$$\hat{n}_2 = -i\partial_\chi \partial_k \hat{n}_1, \qquad \text{etc.}, \tag{39}$$

and sets

$$c(\partial_\chi \psi) = -\partial_T \psi, \tag{40}$$

then one obtains exactly (30) and (31). Thus the solution of (30) and (31) must give a value for c which is equal to (38).

VII. Second-order harmonic

The equations in this order are again formwise identical to those of the first-order harmonics, except that $\omega(k)$ is replaced by $2\omega(k)$. These equations are

$$-2i\omega_e \hat{n}_2^{(2)} + 2ikn_0 \hat{v}_{2x}^{(2)} + \partial_y\left(n_0 \hat{v}_{2y}^{(2)}\right) = S_{2n}^{(2)}, \tag{41a}$$

$$-2i\omega_e \hat{v}_{2x}^{(2)} - 2ik\hat{\phi}_2^{(2)} - \frac{\Delta^2}{\Omega}\hat{v}_{2y}^{(2)} = S_{2x}^{(2)}, \tag{41b}$$

$$-2i\omega_e \hat{v}_{2y}^{(2)} - \partial_y \hat{\phi}_2^{(2)} + \Omega \hat{v}_{2x}^{(2)} = S_{2y}^{(2)}, \tag{41c}$$

$$\partial_y^2 \hat{\phi}_2^{(2)} - 4k^2 \hat{\phi}_2^{(2)} - \hat{n}_2^{(2)} = S_{2\phi}^{(2)}, \tag{41d}$$

where

$$S_{2n}^{(2)} = -2ik\hat{n}_1 \hat{v}_{1x} - \partial_y(\hat{n}_1 \hat{v}_{1y}), \tag{42a}$$

$$S_{2x}^{(2)} = -ik(\hat{v}_{1x})^2 - \hat{v}_{1y}\partial_y \hat{v}_{1x}, \tag{42b}$$

$$S_{2y}^{(2)} = -ik\hat{v}_{1x}\hat{v}_{1y} - \hat{v}_{1y}\partial_y \hat{v}_{1y}, \tag{42c}$$

$$S_{2\phi}^{(2)} = 0. \tag{42d}$$

The only significance of this order is that it contributes to the nonlinearity in third order, and to fully calculate the coefficient of nonlinearity, it is necessary to obtain the solution of this order. The general solution is of the form

$$\hat{v}_{2x}^{(2)} = i\psi^2 \frac{u_{22}}{2k}, \tag{43a}$$

$$\hat{v}_{2y}^{(2)} = \psi^2 \frac{p_{22}}{A_2}, \tag{43b}$$

$$\hat{\phi}_2^{(2)} = i\psi^2\left(\frac{p_{22}\Delta^2}{2A_2k\Omega} - \frac{u_{22}\omega_e}{2k^2}\right) + \psi^2\frac{p^2B/A - u^2}{2k^2}, \tag{43c}$$

$$n_2^{(2)} = ip_{22}\psi^2\left(\frac{4kn_0^2}{\Omega A_2^2} - \frac{\partial_y n_0}{2A_2\omega_e}\right) + \psi^2\frac{2u^2n_0 + 2n_0pu\partial_y(1/A)}{A_2} + p^2\psi^2Q_2, \tag{43d}$$

where

$$Q_2 = \frac{1}{2\omega_e}\partial_y\frac{Q}{A} + \frac{\Omega n_0}{2k\omega_e A_2}\partial_y\left(\frac{B}{A} - \frac{k^2}{A^2}\right)$$
$$- \frac{n_0}{A_2}\left[\frac{2B}{A} + \frac{1}{2\omega_e}\partial_y\left(\frac{Q}{A}\right) - \tfrac{1}{2}\partial_y^2\left(\frac{1}{A^2}\right)\right], \tag{44}$$

$$A_2 = \Omega^2 - 4\omega_e^2, \tag{45a}$$

$$B_2 = \frac{4k^2 - k(\partial_y n_0)/(\omega_e\Omega) - 8k^2\Delta^2 n_0/(\Omega^2 A_2)}{A_2}, \tag{45b}$$

and u_{22} and p_{22} are determined by

$$\partial_y p_{22} - A_2u_{22} = R_{22}, \tag{46a}$$

$$\partial_y u_{22} - B_2p_{22} = \bar{R}_{22}, \tag{46b}$$

where

$$R_{22} = 4i\omega_e u^2 + 4i\omega_e up\partial_y(1/A) - ip^2\partial_y(Q/A)$$
$$-2i\omega_e p^2\left[\frac{2B}{A} - \tfrac{1}{2}\partial_y^2\left(\frac{1}{A}\right)\right] + i\frac{\Omega}{k}p^2\partial_y\left(\frac{k^2}{A^2} - \frac{B}{A}\right), \tag{47a}$$

$$\bar{R}_{22} = 4ik\Delta^2u\frac{u + p\partial_y(1/A)}{\Omega A_2}$$
$$+ ik\Delta^2p^2\frac{\partial_y^2(1/A) - (1/\omega_e)\partial_y(Q/A) - 4B/A}{\Omega A_2}$$
$$+ i\left(n_0 - 4\omega_e^2\right)p^2\frac{\partial_y(k^2/A^2 - B/A)}{\omega_e A_2}. \tag{47b}$$

The boundary conditions are again

$$\hat{\phi}_2^{(2)}(y=0) = 0 = \hat{\phi}_2^{(2)}(y=l). \tag{48}$$

Integration of (46) would require two solutions to be obtained. The boundary condition at the cathode will determine only one of the two possible initial values for $p_{22}(0)$ and $u_{22}(0)$. After integration, when one imposes the boundary condition at the anode, then only one unique linear combination of these two solutions will satisfy both boundary conditions, unless an eigenmode also exists at the harmonic frequency (2ω) and wavevector $(2k)$. In general, this does not occur except accidentally.

VIII. Third order DC

In the case that there are no unstable linear modes, then the background will remain stationary up until this order, which contains the ponderomotive force. The equations are

$$\partial_y\left(n_0 v_{3y}^{(0)}\right) = S_{3y}^{(0)}, \tag{49a}$$

$$-\frac{\Delta^2}{\Omega} v_{3y}^{(0)} = S_{3x}^{(0)}, \tag{49b}$$

$$\Omega v_{3x}^{(0)} - \partial_y \phi_3^{(0)} = S_{3y}^{(0)}, \tag{49c}$$

$$\partial_y^2 \phi_3^{(0)} = S_{3\phi}^{(0)}, \tag{49d}$$

where

$$S_{3n}^{(0)} = -\partial_{\tau'} n_0 - \partial_x\left(n_0 v_{2x}^{(0)}\right) - \partial_x\left(\hat{v}_{1x}^* \hat{n}_1 + \text{c.c}\right)$$
$$- \partial_y\left(\hat{n}_2^* \hat{v}_{1y} + \text{c.c.} + \hat{n}_1^* \hat{v}_{2y} + \text{c.c.}\right), \tag{50a}$$

$$S_{3x}^{(0)} = -\partial_{\tau'} v_0 + \partial_T v_{2x}^{(0)} + \partial_x\left(\phi_2^{(0)} v_0 v_2^{(0)} - \hat{v}_{1x}^* \hat{v}_{1x}\right)$$
$$- v_{1y}^{(0)} \partial_y v_{2x}^{(0)} - \left(\hat{v}_{1y}^* \partial_y \hat{v}_{2x} + \text{c.c.}\right) - \left(\hat{v}_{2y} \partial_y \hat{v}_{1x}^* + \text{c.c.}\right), \tag{50b}$$

$$S_{3y}^{(0)} = -\partial_{\tau'} v_{1y}^{(0)} - \partial_T v_{2y}^{(0)} - v_0 \partial_x v_{2y}^{(0)} - \partial_y\left(v_{1y}^{(0)} v_{2y}^{(0)}\right)$$
$$+ \left(ik \hat{v}_{2y}^* \hat{v}_{1x} + \text{c.c.}\right) + \left(ik \hat{v}_{1y}^* \hat{v}_{2x} + \text{c.c.}\right)$$
$$- \partial_y\left(\hat{v}_{1y}^* \hat{v}_{2y} + \text{c.c.}\right), \tag{50c}$$

$$S_{3\phi}^{(0)} = -\partial^2 \phi_1^{(0)}. \tag{50d}$$

As before, from the above, one obtains the secularity condition (upon integration)

$$\partial_{\tau'}\partial_y\phi_0 + \frac{n_0 v_0}{\Omega}\partial_X v_{2x}^{(0)} + \frac{n_0}{\Omega}\partial_T v_{2x}^{(0)}$$

$$+\Delta^2 C_3 + \frac{\Delta^2}{\Omega^2}\partial_X\int_0^y n_0 v_{2x}^{(0)}\,dy$$

$$+\frac{\Delta^2}{\Omega^2}\partial_X\int_0^y(\hat{v}_{1x}^*\hat{n}_1 + \text{c.c.})\,dy = 0, \tag{51}$$

where C_3 is the constant of integration and is to be determined from the constant voltage condition.

Now consider (51) when there are no unstable linear modes and the background is independent of the slow spatial scale. By (29) and (18)

$$\partial_{\tau'}\partial_y\phi_0 + \Delta^2 C_3 = F_T\partial_T(\psi^*\psi) + F_X\partial_X(\psi^*\psi), \tag{52}$$

where

$$F_T = \frac{2n_0}{\Omega A}\partial_y\left(\frac{p^2}{A}\right), \tag{53a}$$

$$F_X = \frac{2n_0 v_0}{\Omega A}\partial_y\left(\frac{p^2}{A}\right)$$

$$-\frac{2\Delta^2}{\Omega^2}\int_0^y dy\left[\frac{n_0}{A}\partial_y\left(\frac{p^2}{A}\right) + \frac{2upQ}{k}\right]. \tag{53b}$$

Referring back to (36), we see that (52) reduces to

$$\partial_{\tau'}\partial_y\phi_0 + \Delta^2 C_3 = (F_X - cF_T)\partial_X(\psi^*\psi), \tag{54}$$

which gives, upon differentiating,

$$\partial_{\tau'}n_0 + C_3\partial_y n_0 = \partial_X(\psi^*\psi)\cdot\partial_y(F_X - cF_T). \tag{55}$$

The constant C_3 is determined by integrating (52). Thus

$$C_3\int_0^l\Delta^2\,dy = \left[\partial_X(\psi^*\psi)\right]\int_0^l(F_X - cF_T)\,dy. \tag{56}$$

At the moment, it is not possible to discuss the general nature of the functions F_T and F_X, and their integrals, as in (56). However, that is not essential for describing the nature of (55). This equation describes the motion of the background density due to an rf wave. It occurs on the time scale τ' ($= \varepsilon^3 t$) and is slower than all other effects described so far. What is important here is the fact that the evolution of n_0 is driven by $\partial_X(\psi^*\psi)$. This is the ponderomotive force (up to a constant factor). In one region it will drive the density to lower values, while in another region where $\partial_X(\psi^*\psi)$ has the opposite sign, it will force the density to higher values. Of course, as this occurs and as the background density undergoes variations with respect to the slow spatial scale, the first-order DC equations do then key in and become active. Consequently, it is of little use to pursue this expansion beyond third order, because at this order, the first-order DC equations will always become keyed in, and will thereafter dominate, being of a lower order.

IX. Third-order fundamental

This order will give us the nonlinear Schrödinger equation (NLS) [17–19]. The equations are

$$-i\omega_e\hat{n}_3 + ikn_0\hat{v}_{3x} + \partial_y\left(n_0\hat{v}_{3y}\right) = S_{3n}^{(1)}, \tag{57a}$$

$$-i\omega_e\hat{v}_{3x} - ik\hat{\phi}_3 - \frac{\Delta^2}{\Omega}\hat{v}_{3y} = S_{3x}^{(1)}, \tag{57b}$$

$$-i\omega_e\hat{v}_{3y} - \partial_y\hat{\phi}_3 + \Omega\hat{v}_{3x} = S_{3y}^{(1)}, \tag{57c}$$

$$\left(\partial_y^2 - k^2\right)\hat{\phi}_3 - \hat{n}_3 = S_{3\phi}^{(1)}, \tag{57d}$$

where the sources are

$$\begin{aligned} S_{3n}^{(1)} = {} & c\partial_X\hat{n}_2 - \partial_\tau\hat{n}_1 - \partial_X(v_0\hat{n}_2) - \partial_X(n_0\hat{v}_{2x}) \\ & - ik\left(\hat{n}_1 v_{2x}^{(0)} + \hat{n}_1^*\hat{v}_{2x}^{(2)} + \hat{v}_{1x}^*\hat{n}_2^{(2)}\right) \\ & - \partial_y\left(\hat{n}_1 v_{2y}^{(0)} + v_y^{(0)}\hat{n}_2 + \hat{n}_1^*\hat{v}_{2y}^{(2)} + \hat{v}_{1y}^*\hat{n}_2^{(2)}\right), \end{aligned} \tag{58a}$$

$$\begin{aligned} S_{3x}^{(1)} = {} & c\partial_X\hat{v}_{2x} - \partial_X(v_0\hat{v}_{2x}) - ik\left(v_{2x}^{(0)}\hat{v}_{1x} + \hat{v}_{1x}^*\hat{v}_{2x}^{(2)}\right) \\ & - \partial_\tau\hat{v}_{1x} + \partial_X\hat{\phi}_2 - \hat{v}_{2y}^{(2)}\partial_y\hat{v}_{x1}^* - \hat{v}_{2y}^{(0)}\partial_y\hat{v}_{1x} \\ & - \hat{v}_{1y}\partial_y v_{2x}^{(0)} - v_{1y}^{(0)}\partial_y\hat{v}_{2x} - \hat{v}_{1y}^*\partial_y\hat{v}_{2x}^{(2)}, \end{aligned} \tag{58b}$$

$$\begin{aligned} S_{3y}^{(1)} = {} & c\partial_X\hat{v}_{2y} - v_0\partial_X\hat{v}_{2y} - \partial_\tau\hat{v}_{1y} - 2ik\hat{v}_{1x}^*\hat{v}_{2y}^{(2)} \\ & - ik\left(v_{2x}^{(0)}\hat{v}_{1y} - \hat{v}_{2x}^{(2)}\hat{v}_{1y}^*\right) - \hat{v}_{1x}^*\partial_X v_{1y}^{(0)} \\ & - \partial_y\left(v_{2y}^{(0)}\hat{v}_{1y} + \hat{v}_{1y}^*\hat{v}_{2y}^{(2)} + v_{1y}^{(0)}\hat{v}_{2y}\right), \end{aligned} \tag{58c}$$

$$S_{3\phi}^{(1)} = -2ik\partial_X\hat{\phi}_2 - \partial_X^2\hat{\phi}_1. \tag{58d}$$

Before continuing, we shall make some simplifying assumptions. First, we assume the background to be independent of the slow variables, and we take all modes to be stable. Hence ω is real. Then by (9) and (29)

$$v_{1y}^{(0)} = 0 = v_{2y}^{(0)}. \tag{59}$$

We shall also ignore all harmonic terms such as $\hat{v}_{2x}^{(2)}$ etc. We further simplify (57) and (58) by factoring out the dispersive part. This is done by differentiating the first-order equations (15) twice with respect to both k and χ, using (38)–(40). Then, upon comparison with (58) and (59), one observes that the third-order solution is simply

$$\hat{n}_3 = -\tfrac{1}{2}\partial_\chi^2 \partial_k^2 \hat{n}_1 + \delta\hat{n}_3, \qquad \text{etc.,} \tag{60}$$

where the δ-quantities satisfy

$$-i\omega_e \delta\hat{n}_3 + ikn_0 \delta\hat{v}_{3x} + \partial_y(n_0 \delta\hat{v}_{3y}) = \delta S_{2n}^{(1)}, \tag{61a}$$

$$-i\omega_e \delta\hat{v}_{3x} - ik\,\delta\hat{\phi}_3 - \frac{\Delta^2}{\Omega}\delta\hat{v}_{3y} = \delta S_{3x}^{(1)}, \tag{61b}$$

$$-i\omega_e \delta\hat{v}_{3y} - \partial_y \delta\hat{\phi}_3 + \Omega\,\delta\hat{v}_{3x} = \delta S_{3y}^{(1)}, \tag{61c}$$

$$\left(\partial_y^2 - k^2\right)\delta\hat{\phi}_3 - \delta\hat{n}_3 = \delta S_{3\phi}^{(1)}, \tag{61d}$$

and the δ-sources are now only

$$\delta S_{3n}^{(1)} = -\mathscr{D}\hat{n}_1, \tag{62a}$$

$$\delta S_{3x}^{(1)} = -\mathscr{D}\hat{v}_{1x} - \hat{v}_{1y}\partial_y v_{2x}^{(0)}, \tag{62b}$$

$$\delta S_{3y}^{(1)} = -\mathscr{D}\hat{v}_{1y}, \tag{62c}$$

$$\delta S_{3\phi}^{(1)} = 0, \tag{62d}$$

where the operator $\mathscr{D}$ is

$$\mathscr{D} = \partial_\tau + ikv_{2x}^{(0)} - \frac{i}{2}\left(\frac{d^2\omega}{dk^2}\right)\partial_\chi^2, \tag{63}$$

and from (29) and the above

$$v_{2x}^{(0)} = -\psi^*\psi \frac{2}{A\Omega}\partial_y\left(\frac{p^2}{A}\right). \tag{64}$$

All the remaining nonlinear terms only involve $v_{2x}^{(0)}$, which is the second-order shift in the DC shear flow.

To obtain the NLS, we first obtain the equations for $\delta\hat{v}_{3x}$ and $\delta\hat{v}_{3y}$. From (61) and (62),

$$\partial_y(A\,\delta\hat{v}_{3y}) + ikA\,\delta\hat{v}_{3x} = -i\omega_e\,\delta D_3^{(1)} - \Omega\,\delta C_3^{(1)}, \tag{65a}$$

$$\partial_y(-ik\,\delta\hat{v}_{3x}) - AB\,\delta\hat{v}_{3y} = -i\frac{\Delta^2 k}{\Omega A}\delta D_3^{(1)} + k\frac{n_0-\omega_e^2}{A\omega_e}\delta C_3^{(1)}, \tag{65b}$$

where

$$\delta D_3^{(1)} = \partial_y\,\delta S_{3y}^{(1)} + ik\,\delta S_{3x}^{(1)} + \frac{i}{\omega_e}\delta S_{3x}^{(1)} + \delta S_{3\phi}^{(1)}, \tag{66a}$$

$$\delta C_3^{(1)} = \partial_y\,\delta S_{3x}^{(1)} - ik\,\delta S_{3y}^{(1)}. \tag{66b}$$

Now, as a consequence of (20) and (65), we have

$$\begin{aligned}\partial_y(Au\,\delta\hat{v}_{3y} + ikp\,\delta\hat{v}_{3x})\\ = i\left(p\frac{\Delta^2 k}{A\Omega} - \omega_e u\right)\delta D_3^{(1)}\\ -\left(\Omega u + kp\frac{n_0-\omega_e^2}{A\omega_e}\right)\delta C_3^{(1)}.\end{aligned} \tag{67}$$

Lastly, integrating (67) from the anode to the cathode will give the NLS. One must use the boundary conditions (17) and

$$\delta\hat{\phi}_3(y=0) = 0 = \delta\hat{\phi}_3(y=l) \tag{68}$$

along with (61b). One then reduces (67) to

$$\tilde{N}\left(i\partial_\tau + \frac{1}{2}\frac{d^2\omega}{dk^2}\partial_x^2\right)\psi + N\psi^*\psi^2 = 0, \tag{69}$$

where one can show that

$$\tilde{N} = \int_0^l dy\frac{p(\partial_y n_0)}{A\omega_e}\left(u - \frac{k\Omega p}{A\omega_e}\right) - \frac{n_0 upA}{\omega_e}\bigg|_{y=0}^{l}, \tag{70a}$$

$$N = +\frac{2p^2k^2n_0}{A^3\omega_e^2}\partial_y\left(\frac{p^2}{A}\right)\bigg|_{y=0}^{l} - \frac{2}{\Omega}\int_0^l dy\frac{k^3}{A}\mathcal{N}\partial_y\left(\frac{p^2}{A}\right). \tag{70b}$$

with $\mathcal{N}$ given by

$$\mathcal{N} = \frac{p^2}{A^2}\left(\frac{\Omega\partial_y n_0}{k\omega_e^2} - \frac{4n_0^2}{A\omega_e} + \frac{2n_0}{\Omega}\right) + \frac{2n_0}{\omega_e^3}\left(\frac{n_0 p}{A} + \frac{\omega_e}{k}u\right)^2. \tag{70c}$$

Now we have $\tilde{N}$ and N given entirely in terms of the background quantities and the first-order solution.

From this form, one could numerically evaluate these coefficients. However, from the general structure it is possible for us to make certain general statements. Consider first the $\tilde{N}$-term in (70a). There we see that the integrand is proportional to $\partial_y n_0$. Thus the integrand has a nonzero contribution only along a density gradient. Furthermore, unless ω_e or A has a zero inside the density gradient (thereby giving a logarithmic contribution to the integral), the quantity $\tilde{N}$ will be real for real ω.

On the other hand, the second term, N, will in general be complex if A or ω_e ever vanishes inside the plasma sheath. As one can see from (70c), there are apparent singular terms whenever ω_e or A has a zero anywhere inside the sheath. Thus, whenever $\mathcal{N}$ has a singular point inside the plasma sheath, a complex value for N will in general result.

When this is the case, (69) reduces to a nonlinear Schrödinger equation with a complex coefficient, namely

$$\partial_\tau q + \frac{1}{2}\frac{d^2\omega}{dk^2}\partial_x^2 q + \Gamma\psi^*\psi^2 = 0, \tag{71}$$

where Γ ($= N/\tilde{N}$) is complex in general. If the imaginary part of Γ were negative, then a nonlinear instability would occur (while if the imaginary part were positive, a nonlinear stabilization would occur).

X. Summary

As we have detailed here, the lowest-order instability is the first-order DC, which is yet to be analyzed. However, because this is a long-wavelength instability, it is not expected to be of any importance until some lower-order instability has keyed it in. In particular, the second-order DC may key it in, on the second-order time scale, if there is a linear instability. And even if nothing else is unstable, then at least, after the third-order time scale, the third-order DC equations (ponderomotive) will definitely key it in. However, before this last occurs, one would expect the third-order fundamental equations to go unstable by the nonlinear instability mentioned in Section IX. This instability will be on the second-order time scale, and thus faster than that of the third-order DC.

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Linear stability of Vlasov–Poisson electron plasma in crossed fields. Perturbations propagating parallel to the magnetic field

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It is shown that electrostatic Vlasov–Poisson perturbations that propagate parallel to the magnetic field in a planar magnetron are stable for both an isotropic and also for a particular anisotropic ($T_y = 3T_x$) temperature distribution. The inhomogeneity of the electron density is fully incorporated in the analysis. The proof makes use of only the dispersion relation of Trivelpiece–Gould type, without actually solving the eigenvalue equation. These results suggest, not unexpectedly, that these modes should be stable for all such anisotropic velocity distributions.

1. Introduction

The stability of electron plasmas in crossed fields has been the subject of wide investigation in connection with the operation of such devices as magnetrons, diodes and crossed-field amplifiers (Buneman, Levy & Linson 1966; Thomas 1982; Chernin & Lau 1984; Davidson & Chang 1984, 1985; Ott *et al.* 1985). The diocotron instability (Knauer 1966; Davidson 1974), among others, is the most familiar instability encountered in the cross-field configuration; it is a longitudinal (electrostatic) instability with the wave vector **k** perpendicular to the direction of the operating magnetic field so that the electron drift velocity $\mathbf{v}_0$ is parallel to the **k** vector.

Non-neutral plasmas in the crossed-field configuration are inherently inhomogeneous in the density distribution, and the velocity distribution is often found to be anisotropic, so that the plasmas can be characterized by the different temperatures along appropriate directions (Kaup, Hansen & Thomas 1985). A non-Maxwellian energy distribution of the plasma particles can often give rise to electrostatic instabilities, so it is of interest to study the possible instabilities of such a distribution. The purpose of this work is to investigate the possibility of any unstable longitudinal waves propagating parallel to the magnetic field in an inhomogeneous and anisotropic electron plasma as would exist in a magnetron or related microwave device.

Although it has been asserted that a parallel component of the wave vector tends to reduce the growth rate, thus stabilizing an otherwise unstable wave

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(Buneman *et al.* 1966; Crawford 1967), there has been no specific analysis of the high-density ($\omega_p^2 \approx \Omega_c^2$), inhomogeneous and anisotropic regimes that one would expect to exist in a microwave device. Thus there is a lack of understanding of how a parallel component would tend to stabilize an unstable wave, particularly for the Vlasov–Poisson system and for typical inhomogeneous and anisotropic distributions that could occur in the crossed-field configuration.

In this paper we provide such an analysis, showing that, even with a moderately strong anisotropic distribution, instabilities associated with longitudinal waves propagating parallel to the magnetic field do not occur. In particular, we find that certain general features of the plasma dispersion function may be used to demonstrate the stability of these modes.

In homogeneous plasmas the propagation of waves subject to spatial boundary conditions is characterized by an eigenvalue equation. Our proof only requires the use of an effective dispersion relation of Trivelpiece–Gould (1959) type, without actually solving the eigenvalue equation.†

This paper is organized as follows. In §2 the basic equations are introduced and the equilibrium distribution function is discussed. The general procedure leading to the eigenvalue equation is also described. In §3 the linearized solution of the Vlasov equation is obtained for an anisotropic equilibrium distribution function with three different temperatures. Differential equations describing the propagation of the electrostatic waves are examined in various limiting cases. Finally, we prove the stability of the wave, subject to the boundary conditions of a smooth-bore planar magnetron, for the isotropic case and then also for a moderately strong ($T_y = 3T_x$) anisotropic case. The results for these two different cases are identical. Considering the general case, we then argue that these results strongly suggest that this mode will be stable for all similar velocity distributions.

2. Basic equations and particle orbit in unperturbed fields

To describe our electron plasma in a planar magnetron (see figure 1), we use the Vlasov and Poisson equations

$$\frac{\partial f}{\partial t} + \mathbf{v}.\nabla f - \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}_0\right).\nabla_v f = 0, \tag{1}$$

$$\nabla.\mathbf{E} = -4\pi e\int f\,d^3v, \tag{2}$$

where the notations follow standard usage in plasma physics. The magnetic field is taken to be entirely external and constant in time and space, and we take the perturbations to be purely electrostatic. We shall designate by a subscript zero the equilibrium (zeroth-order) quantities, which are the steady-state solutions of (1) and (2):

$$(\mathbf{v}.\nabla)f_0 - \frac{e}{m}\left(\mathbf{E}_0 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_0\right).\nabla_v f_0 = 0, \tag{3}$$

$$\nabla.\mathbf{E}_0 = -4\pi e\int f_0\,d^3v. \tag{4}$$

The relative directions of the crossed fields $\mathbf{E}_0$ and $\mathbf{B}_0$ are illustrated in figure 1.

† Strictly speaking, we only prove 'spectral stability', and not 'linear stability'. For a discussion of these two definitions and this point, see Holm *et al.* (1985).

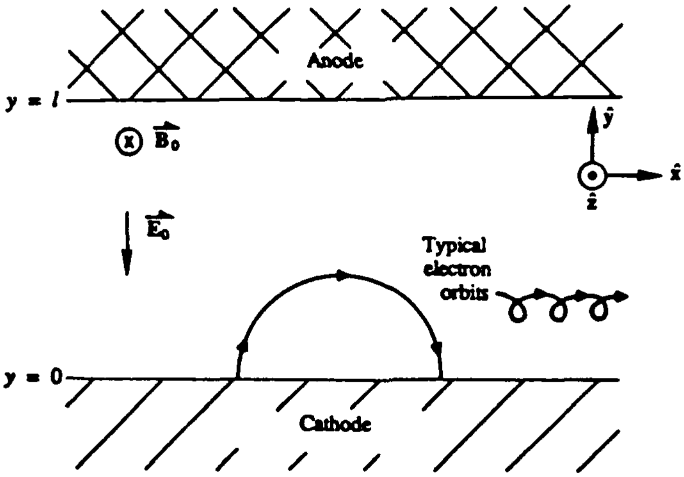

FIGURE 1. Geometry and electron orbits in the planar magnetron.

It is well known that (3) is satisfied by any f_0 that is a function of only the constants of the motion of the particle in the unperturbed fields $\mathbf{E}_0$ and $\mathbf{B}_0$. We have the following three constants of motion (Kaup *et al.* 1985)

$$p_x = m(v_x - \Omega v), \tag{5a}$$

$$p_z = mv_z, \tag{5b}$$

$$w = \tfrac{1}{2}m(v_x^2+v_y^2+v_z^2) - e\Phi_0(y), \tag{5c}$$

where $\Omega = eB_0/mc$ is the gyrofrequency and Φ_0 is the electric potential associated with the equilibrium electric field $\mathbf{E}_0$ ($\mathbf{E}_0 = -\nabla\Phi_0$).

Thus $f_0(p_x, p_z, w)$ satisfies (3), and the equilibrium density $n_0 = \int f_0 \, d^3v$ is determined by (4) once the functional form of $f_0(p_x, p_z, w)$ has been given.

Now (1) and (2) are linearized with respect to the perturbed quantities $f' = f - f_0$ and $E' = E - E_0$ to obtain

$$\frac{\partial f'}{\partial t} + \mathbf{v}.\nabla f' - \frac{e}{m}\left(\mathbf{E}_0 + \frac{\mathbf{v}}{c}\times\mathbf{B}_0\right).\nabla_{\mathbf{v}} f' = -\frac{e}{m}\nabla\Phi'.\nabla_{\mathbf{v}} f_0, \tag{6}$$

$$\nabla^2\Phi' = 4\pi e \int f' \, d^3v \qquad (\mathbf{E}' = -\nabla\Phi') \tag{7}$$

Equation (6) can be solved by integration along the unperturbed orbit:

$$f(\mathbf{x}, \mathbf{v}, t) = -\frac{e}{m}\int_{-\infty}^{t} dt' \, \nabla'\Phi(\mathbf{x}', t').\nabla_{\mathbf{v}'} f_0(\mathbf{x}', \mathbf{v}'), \tag{8}$$

where we shall henceforth drop the primes on f' and Φ'. We resort to normal-mode analysis for (7), seeking the solution in the form

$$\Phi(\mathbf{x}, t) = \hat{\Phi}(y)\, e^{ikx - i\omega t},$$

which is appropriate for a wave propagating in the direction parallel to the magnetic field in a medium with inhomogeneity along the y-direction. Then (7) and (8) yield

$$\left(-k^2+4\pi e^2\int d^3v\frac{\partial f_0}{\partial w}+\frac{d^2}{dy^2}\right)\hat{\Phi}(y)$$
$$= -4\pi i e^2\int d^3v\left(\omega\frac{\partial f_0}{\partial w}+k\frac{\partial f_0}{\partial p_z}\right)\int_{-\infty}^{0} d\tau\,\hat{\Phi}(y')\,e^{ik\xi_z-i\omega\tau}, \quad (9)$$

where we have substituted the general functional form of the distribution function $f_0(\mathbf{x}',\mathbf{v}') = f_0(p_x,p_z,w)$; $\tau = t'-t$, and the orbits should satisfy the initial conditions

$$\left.\begin{aligned}\xi_y &= y'-y = 0,\\ \xi_z &= z'-z = 0,\\ \mathbf{v}' &= \mathbf{v}\end{aligned}\right\} \text{ at } \tau = 0, \quad (10)$$

In general, (9) is an integral equation for the potential $\hat{\Phi}$. Instead of solving an integral equation, we choose to expand $\hat{\Phi}(y')$ about the position y:

$$\hat{\Phi}(y') = \hat{\Phi}(y)+\xi_y\frac{d\hat{\Phi}}{dy}+\tfrac{1}{2}\xi_y^2\frac{d^2\hat{\Phi}}{dy^2}. \quad (11)$$

This is certainly a good approximation if the orbit size (gyroradius A) is much smaller than the scale length of the inhomogeneity ($L = |\nabla E_0/E_0|^{-1}$). Kaup *et al.* (1985) calculated single-particle orbits in a spatially varying electric field $\mathbf{E}_0(y)$ and constant magnetic field $\mathbf{B}_0$ by a singular perturbation method, assuming $\epsilon = A/L$ to be small. We write down their orbit, neglecting terms $O(\epsilon^2)$:

$$\left.\begin{aligned}v_x' &= v_0+A\Omega\cos(\Delta\tau+\phi),\\ v_y' &= -A\Delta\sin(\Delta\tau+\phi),\\ \xi_x &= v_0\tau+\frac{A\Omega}{\Delta}[\sin(\Delta\tau+\phi)-\sin\phi],\\ \xi_y &= A[\cos(\Delta\tau+\phi)-\cos\phi],\end{aligned}\right\} \quad (12)$$

where $v_0 = eE_0(y)/m\Omega$ is the drift velocity and $\Delta = (\Omega^2-4\pi e^2 n_0/m)^{\frac{1}{2}} \equiv [\Omega^2-\omega_p^2(y)]^{\frac{1}{2}}$ is the reduced gyrofrequency (Kaup *et al.* 1985; Prasad, Morales & Fried 1985).

In (12) the velocity components at $\tau = 0$ are solved in terms of the constants of integration A and ϕ:

$$v_x = v_0+A\cos\phi, \quad v_y = -\Delta A\sin\phi. \quad (13)$$

Substituting (11) and (12) into (9) and carrying out the time integration, we obtain, with the aid of (13),

$$\left(-k^2+4\pi e^2\int d^3v\frac{\partial f_0}{\partial w}+\frac{d^2}{dy^2}\right)\hat{\Phi}$$
$$= -4\pi e^2\int d^3v\left(\omega\frac{\partial f_0}{\partial w}+k\frac{\partial f_0}{\partial p_z}\right)\left(I_0\hat{\Phi}+I_1\frac{d\hat{\Phi}}{dy}+\tfrac{1}{2}I_2\frac{d^2\hat{\Phi}}{dy^2}\right), \quad (14)$$

where, in the respective lowest orders,

$$I_0 = \frac{1}{kv_z - \omega}, \tag{15a}$$

$$I_1 = \frac{v_x - v_0}{2\Omega}\left(\frac{-2}{kv_z-\omega} + \frac{1}{kv_z - \omega \pm \Delta}\right), \tag{15b}$$

$$I_2 = \frac{(v_x - v_0)^2}{\Omega^2}\left(\frac{\frac{3}{2}}{kv_z-\omega} - \frac{1}{kv_z-\omega\pm\Delta} + \frac{\frac{1}{4}}{kv_z-\omega\pm 2\Delta}\right)$$
$$+\frac{v_y^2}{\Delta^2}\left(\frac{\frac{1}{2}}{kv_z-\omega} - \frac{\frac{1}{4}}{kv_z-\omega\pm 2\Delta}\right). \tag{15c}$$

In (15) terms with double signs ($\pm$) are summed over the two signs.

3. Eigenvalue equation and proof of stability

To proceed further, it is necessary to have a specific equilibrium distribution function $f_0(p_x, p_z, w)$. Kaup *et al.* (1985) devised a model distribution function to model an electron plasma in a crossed-field planar magnetron:

$$f_0(p_x, p_z, w) = Ne^{-\beta w}\exp\left(-\frac{\gamma-\beta}{2m}p_x^2\right)\exp\left(-\frac{\delta-\beta}{2m}p_z^2\right), \tag{16}$$

where w, p_x and p_z are the constants of motion defined by (5) and N is a constant. In terms of the velocity components v_x, v_y and v_z, (16) reads

$$f_0(v_x, v_y, v_z) = NF(y)\exp\left[-\frac{\gamma m}{2}\left(v_x - \frac{\gamma-\beta}{\gamma}\Omega y\right)^2\right]\exp\left(-\frac{\beta m}{2}v_y^2\right)\exp\left(-\frac{\delta m}{2}v_z^2\right), \tag{16'}$$

with
$$F(y) = \exp[\beta e\Phi_0(y)]\exp\left[\frac{\beta}{2\gamma}(\beta-\gamma)\, m\Omega^2 y^2\right]. \tag{17}$$

Then the unperturbed density is calculated as

$$n_0(y) = \int f_0\, d^3v = N\left[\frac{(2\pi)^3}{\beta\gamma\delta m^3}\right]^{\frac{1}{2}} F(y). \tag{18}$$

Substituting (16) into (14) and carrying out the necessary velocity integrals, we obtain, after lengthy algebra,

$$\left\{1 - \omega_p^2\left[\frac{\delta}{2\Omega^2}\left(\frac{1}{\gamma} + mu^2\right)Q + \frac{\delta}{2\beta\Delta^2}R\right]\right\}\frac{d^2\hat{\Phi}}{dy^2} - \frac{mu}{2\Omega}\omega_p^2(y)\,\delta P\frac{d\hat{\Phi}}{dy}$$
$$-\{k^2 + 4\pi e^2 n_0 \delta[1 + \zeta_0 Z(\zeta_0)]\}\hat{\Phi} = 0, \tag{19}$$

where
$$P = -2\zeta_0 Z(\zeta_0) + \sum_{+,-}\left[1 \pm \frac{\beta\Delta}{\delta(\omega \mp \Delta)}\right]\zeta_{\mp 1} Z(\zeta_{\mp 1}), \tag{20a}$$

$$Q = \tfrac{1}{2}\zeta_0 Z(\zeta_0) - \sum_{+,-}\left[1 \pm \frac{\Delta\beta}{\delta(\omega \mp \Delta)}\right]\zeta_{\mp 1} Z(\zeta_{\mp 1}) + \tfrac{1}{2}\sum_{+,-}\left[1 \pm \frac{2\beta\Delta}{\delta(\omega \mp 2\Delta)}\right]\zeta_{\mp 2} Z(\zeta \mp 2), \tag{20b}$$

$$R = \tfrac{1}{2}\zeta_0 Z(\zeta_0) - \tfrac{1}{4}\sum_{+,-}\left[1 \pm \frac{2\beta\Delta}{\delta(\omega \mp 2\Delta)}\right]\zeta_{\mp 2} Z(\zeta_{\mp 2}), \tag{20c}$$

$$\zeta_{\pm n} = \frac{\omega \pm n\Delta}{k}\left(\frac{m\delta}{2}\right)^{\frac{1}{2}} \quad (n = 0, 1, 2), \tag{21a}$$

$$Z(\zeta) = \frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} \frac{e^{-t^2}}{t - \zeta}\delta\tau, \tag{21b}$$

which is the plasma dispersion function (Fried & Conte 1961), and

$$u = \frac{\gamma - \beta}{\gamma}\Omega y - \frac{eE_0}{m\Omega} = -\frac{1}{m\Omega\beta}\frac{1}{n_0}\frac{dn_0}{dy}, \tag{21c}$$

which is the Larmor drift velocity. It is legitimate to neglect mu^2 in (19) as compared with $1/\gamma$, since the Larmor drift velocity is much smaller than the thermal velocity (v_T) because of the assumed smallness of $A/L\,(u/v_T = (v_T/n_0)\,dn_0/dy = A/L \ll 1)$. This neglect is even better justified in the cold-fluid limit.

We shall now consider a few limiting cases of (19).

(i) *Cold-plasma limit* ($\omega/k \gg v_T$)

Expanding the plasma dispersion function in powers of $(k^2/\omega^2)\,v_T^2$ and taking the limit as $v_T \to 0$, we can reduce (20) to

$$P = -\frac{2\beta\Delta^2}{\delta(\omega^2 - \Delta^2)},$$

$$Q = \frac{2\beta\Delta^2}{\delta}\left(\frac{1}{\omega^2 - \Delta^2} - \frac{1}{\omega^2 - 4\Delta^2}\right),$$

$$R = \frac{2\beta\Delta^2}{\delta}\frac{1}{\omega^2 - 4\Delta^2}.$$

Substitution of these expressions into (19) yields

$$\frac{d^2\hat{\Phi}}{dy^2} - \frac{d\omega_p^2/dy}{\omega^2 - \Omega^2 + \omega_p^2}\frac{d\hat{\Phi}}{dy} - k^2\left(1 - \frac{\omega_p^2}{\omega^2}\right)\frac{\omega^2 - \Omega^2 + \omega_p^2}{\omega^2 - \Omega^2}\hat{\Phi} = 0. \tag{22}$$

In obtaining (22), we have put $\gamma = \beta$ and $\Delta^2 \approx \Omega^2$. Equation (22) could have been obtained by starting with the cold fluid equations.

(ii) *Massless guiding-centre limit*

If the electrons are treated as a massless guiding-centre fluid ($m \to 0$) then P, Q and R all vanish, to yield

$$\frac{d^2\hat{\Phi}}{dy^2} - k^2\hat{\Phi} - 4\pi e^2 n_0\,\delta[1 + \zeta_0 Z(\zeta_0)]\,\hat{\Phi} = 0. \tag{23}$$

This equation could have been obtained by employing the drift kinetic approximation, in which the electron dynamics in the plane perpendicular to

$\mathbf{B}_0$ are suppressed. We can also see that the drift approximation is consistent with the approximation of very high frequency, $\omega \gg \Delta$. In this case $\zeta_\pm \approx \zeta_0$, and P, Q and R all become zero.

(iii) *Low-frequency wave*: $\Delta^2 = \Omega^2 - \omega_p^2 \gg \omega^2$

When $\Delta^2 \gg \omega^2$ and $\beta \approx \zeta$, the terms indicated by $\Sigma_{+,-}$ in the expressions for P, Q and R in (20) are negligibly small compared with $\zeta_0 Z(\zeta_0)$. Equation (19) then becomes

$$\frac{d^2\hat{\Phi}}{dy^2}\left[1-\frac{\omega_p^2}{4}\zeta_0 Z(\zeta_0)\left(\frac{3\beta}{\gamma\Omega^2}+\frac{1}{\Delta^2}\right)\right]-\frac{\zeta_0 Z(\zeta_0)}{\Omega^2}\frac{d\omega_p^2}{dy}\frac{d\hat{\Phi}}{dy}$$
$$-\{k^2+4\pi e^2 n_0\,\delta[1+\zeta_0 Z(\zeta_0)]\}\,\hat{\Phi}=0,$$

which can be cast into the adjoint form

$$\frac{d}{dy}\left[(1-\alpha\omega_p^2)^\nu\frac{d\hat{\Phi}}{dy}\right]-\{k^2+4\pi e^2 n_0[1+\zeta_0 Z(\zeta_0)]\}\,(1-\alpha\omega_p^2)^{\nu-1}\,\hat{\Phi}=0, \tag{24}$$

where

$$\alpha=\frac{\zeta_0 Z(\zeta_0)}{\Omega^2\nu}, \qquad \nu=\frac{4}{3\beta/\gamma+1}. \tag{25}$$

Regardless of which case we are considering, in order to determine $\hat{\Phi}$ uniquely from one of the above differential equations, it is still necessary to specify two boundary conditions. We shall consider the two parallel planes $y=0$ and $y=l$ to be perfect conductors, as in a planar magnetron. Then the boundary conditions are that the Fourier amplitude $\hat{\Phi}$ must vanish at the two boundary planes:

$$\hat{\Phi}(0)=\hat{\Phi}(l)=0. \tag{26}$$

Next, an effective dispersion relation of Trivelpiece–Gould type, which is the solvability condition for the differential equation subject to the boundary condition, is obtained by operating with $\int_0^l \Phi^* dy$ (* denotes the complex conjugate) on the differential equation. If we perform this operation on (24), first assuming $\nu=1$ ($\beta=\gamma$), then we obtain

$$\int_0^l dy\left[1+\frac{\omega_p^2}{\Omega^2}(1+\tfrac{1}{2}Z')\right]\left|\frac{d\hat{\Phi}}{dy}\right|^2+\int_0^l(k^2-2\pi e^2 n_0\,\delta Z')\,|\hat{\Phi}|^2\,dy=0, \tag{27}$$

where

$$Z'=\frac{dZ(\zeta_0)}{d\zeta_0}=-2[1+\zeta_0 Z(\zeta_0)].$$

Putting $Z'=Z'_r+iZ'_i$, where Z'_r and Z'_i represent respectively the real and imaginary parts of Z', (27) separates into real and imaginary parts:

$$\int_0^l dy\left[\left(1+\frac{\omega_p^2}{\Omega^2}\right)\left|\frac{d\hat{\Phi}}{dy}\right|^2+k^2|\hat{\Phi}|^2\right]+Z'_r\int_0^l dy\left(\frac{\omega_p^2}{2\Omega^2}\left|\frac{d\hat{\Phi}}{dy}\right|^2-2\pi e^2 n_0\,\delta|\hat{\Phi}|^2\right)=0, \tag{28}$$

$$Z'_i\left(\int_0^l dy\,\frac{\omega_p^2}{2\Omega^2}\left|\frac{d\hat{\Phi}}{dy}\right|^2-\int_0^l 2\pi e^2 n_0\,\delta|\hat{\Phi}|^2\,dy\right)=0. \tag{29}$$

Equation (29) indicates that the solvability condition is that either Z'_i or the quantity in parentheses be zero. However, the latter condition makes it impossible to satisfy (28), since the first term of (28) is positive-definite and the last term would have to vanish. Therefore the solvability condition for the system (24) subject to (26) is

$$Z_i' = \mathrm{Im}\left[\frac{dZ(\zeta_0)}{d\zeta_0}\right] = 0. \tag{30}$$

We can see that (30) is also the solvability condition for the system (23) and (26), observing that (23) is formally obtained from (24) by taking the limit as $\Omega \rightarrow \infty$. If an exactly parallel procedure is carried out for the cold-fluid equation (22) then the solvability condition turns out to be $\mathrm{Im}\,(\omega) = 0$, which is the cold-fluid limit of (30). Thus electrostatic parallel waves in the cold-fluid approximation are stable in the planar magnetron.

Returning to (30), we examine the possibility that it could be satisfied with $\mathrm{Im}\,(\omega) > 0$. The tabulation of the plasma dispersion function (Fried & Conte 1961) indicates that the only points on the complex ζ_0 plane that satisfy $Z_i' = 0$ with $\mathrm{Im}\,(\omega) > 0$ are those points on the positive $\mathrm{Im}\,(\zeta_0)$ axis. But the value of Z_r' on the positive $\mathrm{Im}\,(\zeta_0)$ axis is restricted by $-2 < Z_r' < 0$. However, we can immediately see that with $-2 < Z_r' < 0$, (28) cannot be satisfied, since it can now be reduced to a series of positive-definite terms. In summary, the eigenvalues of the systems (24) (with $\nu = 1$) and (26) should be selected from those points satisfying $Z_i' = 0$ in the lower-half ζ_0 plane. These eigenmodes are therefore only Landau-damped.

It is a foregone conclusion that the eigenvalues of (23) will also give only damping as observed in the solvability condition (28) when we take the limit $\Omega \rightarrow \infty$.

Finally, we consider one case of anisotropic temperature by assuming $\nu = 2$ ($\gamma = 3\beta$ or $T_y = 3T_x$) in (24). The analysis of the case for a general non-integral value of ν is much more difficult, but the following result for $\nu = 2$ strongly suggests that the above does extend to general values of ν. Similar algebra to that employed above yields

$$\alpha_i\left\{\int_0^l 2\omega_p^2(\alpha_r\omega_p^2 - 1)\left|\frac{d\hat{\Phi}}{dy}\right|^2 dy + \int_0^l [4\pi e^2 n_0\,\delta(2\Omega^2 - 4\Omega^2\omega_p^2\alpha_r - \omega_p^2) - k^2\omega_p^2]\,|\hat{\Phi}|^2\,dy\right\} = 0, \tag{31}$$

$$\int_0^l [1 - 2\alpha_r\omega_p^2 + \omega_p^4(\alpha_r^2 - \alpha_i^2)]\left|\frac{d\hat{\Phi}}{dy}\right|^2 dy + \int_0^l \{4\pi e^2 n_0\,\delta[1 - \alpha_r\omega_p^2 + 2\alpha_r\Omega^2 - 2\Omega^2\omega_p^2(\alpha_r^2 - \alpha_i^2)] + k^2(1 - \alpha_r\omega_p^2)\}\,|\hat{\Phi}|^2\,dy = 0, \tag{32}$$

where

$$\alpha_r = -\frac{1}{2\Omega^2}(1 + \tfrac{1}{2}Z_r'), \qquad \alpha_i = -\frac{1}{4\Omega^2}Z_i'.$$

Again, as a solvability condition, either Z_i' or the quantity in braces in (31) should vanish. If we choose the latter condition then (31) and (32) combine to give

$$\int_0^l\left[\left|\frac{d\hat{\Phi}}{dy}\right|^2 + (k^2 + 4\pi e^2 n_0\,\delta)\,|\hat{\Phi}|^2\right]dy + (\alpha_r^2 + \alpha_i^2)\int_0^l \omega_p^4\left(\frac{4\Omega^2}{v_T^2}|\hat{\Phi}|^2 - \left|\frac{d\hat{\Phi}}{dy}\right|^2\right)dy = 0. \tag{33}$$

In our present theory the last term of (33) can never be negative since the ratio of $(\Omega^2/v_T^2)|\Phi|^2$ to $|d\Phi/dy|^2$ is $O(A^2/L^2)$. Thus (33) cannot be satisfied, and one of the solvability conditions for the anisotropic distribution must again be $Z_i' = 0$. The possibility of growing eigenvalues is rejected since the other solvability condition (32) with $\alpha_i = 0$ can again never be satisfied if $-2 < Z_r' < 0$, which gives $\alpha_r < 0$. In conclusion, even for a distribution function with anisotropic temperatures ($\nu = 2$), the eigenmodes are subject to only Landau damping.

In the general case of non-integral values of ν, it is certainly true that $\alpha_i = 0$ is a solution. However, it cannot readily be verified that this is the only possible solution for the imaginary part of the Trivelpiece–Gould dispersion relation, owing to the occurrence of irrational powers in (24). Thus we cannot at present exclude the possibility of the existence of some other unstable mode where $\alpha_i \neq 0$. However, we do not expect any such modes to occur, since no such mode was present for the $\nu = 2$ case. We shall assume that no such mode exists. Thus instability could only occur for $\alpha_i = 0$, and again we have $\alpha_r < 0$ and the real part of the Trivelpiece–Gould dispersion relation being positive-definite. Thus unstable modes with $\alpha_i = 0$ cannot exist, and the eigenmode must be Landau-damped whenever $\alpha_i = 0$. The total exclusion of the possibility of any other modes ($\alpha_i \neq 0$) would probably require an analysis with numerical or analytical solutions of (24).

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Two-dimensional nonlinear Schroedinger equation and self-focusing in a two-fluid model of Newtonian cosmological perturbations

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Summary. We study the evolution of weakly nonlinear hydrodynamic disturbances on a static cosmological background, paying special attention to nonlinear modulational instabilities, solitons, and self-focusing. Our model consists of two distinct nonrelativistic fluid components coupled only by gravitation. The two-dimensional (cubic) Nonlinear Schroedinger Equation (NLS) is found to govern the long-term evolution of the envelope of weakly nonlinear, nearly plane-symmetric, almost monochromatic acoustic waves. Nonlinear modulational instability may even arise within a range of wavenumbers for which both modes of the linearized theory are (Jeans-) stable, leading to the possibility of soliton formation. Extrapolated to a realistic expanding universe, this result suggests that nonlinear modulational instability might "switch on" before the linear (Jeans) instability. Moreover – in contrast to the one-fluid case – the two-fluid system studied here also exhibits a violent nonlinear self-focusing instability of the type observed in experiments with optical beams. Provided certain restrictions on the wavenumber and initial conditions are satisfied, self-focusing leads to a steep rise in the density contrast at certain isolated points in two dimensions, corresponding to lines in three dimensions. (Of course, the present theory can only follow the evolution of self-focusing singularities until the condition of weak nonlinearity is violated.) We also find some evidence for the onset of nonlinear saturation of linear Jeans instabilities. If present, saturation would imply that nonlinear instabilities might dominate, at least under certain circumstances. In that case, the usual picture of biased galaxy formation at the Jeans mass scale would have to be regarded as oversimplified. Independently of our particular model, the possible existence of nonlinear modulational instability implies that caution should be excercised when interpreting the results of certain "*n*-body" numerical simulations used by various researches in large-scale structure and galaxy formation: Any numerical method which (in effect) filters out the high-wavenumber part of the initial fluctuation spectrum may seriously underestimate the effects of resonant phenomena which can transfer power from high to low wavenumbers.

Key words: galaxies: formation – cosmology – gravitation

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1. Introduction and model

It is becoming increasingly difficult to reconcile standard models for the formation of large-scale structures and galaxies with upper limits on the small-scale anisotropy of the microwave background (Uson and Wilkinson, 1985). Recent observations of large-scale structures such as filaments and voids, together with the task of explaining fundamental properties of galaxies (such as their characteristic masses), have led recently to exotic proposals, such as the existence of cosmic strings. Without taking a position on the merits of such exotic proposals, we investigate in this paper the possibility that one can do without them. We will see that acoustic waves in a self-gravitating system of two decoupled nonrelativistic fluids may suffer nonlinear (modulational) instability and may form self-focusing singularities, depending on the relative densities, the wavelength, and the local initial conditions.

The predictions made by any linear theory of fluctuations strongly depend on assumptions made about the "initial" fluctuations. In contrast, nonlinear systems often exhibit qualitatively similar behaviour for large classes of initial conditions. The fact that nonlinear effects near recombination are presumably still weak does not necessarily preclude strong qualitative deviations from the predictions of linear theories, provided the evolution time is sufficiently long for the nonlinear effects to accumulate (Tomita, 1967, 1971; Juszkiewicz, 1981; Peebles, 1970; Liang, 1976, 1977; Vishnaic, 1982). By means of a multiple-time-scales approach, we will see that linear dispersion and cubic nonlinearity may have comparable effects over sufficiently long time scales. The evolution of a wave envelope with "slow" transverse dependence will be seen to satisfy the well-known Nonlinear Schroedinger equation in two dimensions.

A nonlinear mechanism for the (occasional) formation of filamentary structures by nonlinear self-focusing was recently suggested by one of us (Kates, 1986). It was erroneously claimed that self-focusing occurs (for certain range of the wavenumber) in a model consisting of a *single* fluid on a static cosmological background. The correct coefficients for the one-fluid NLS equation will be given in the Appendix of this paper. As it turns out, if only one fluid is present, modulational instability can indeed occur, as claimed, but the coefficients of the resulting NLS equation never admit self-focusing singularities, because the linear dispersion term does not change sign. In the present two-fluid model, the linear dispersion term can take both signs.

Our interest in studying a system containing two fluids is also motivated by strictly astrophysical considerations. Recently, considerable evidence has been found in support of a dark-matter hypothesis: suppose the predominant constituent of the universe since recombination has been some form of dark matter that does not interact directly with ordinary matter and in particular with electromagnetic radiation, except through the gravitational potential. If one also supposes that "biasing" occurs, it is possible to avoid gross contradictions with the observations (Frieman and Will, 1982; Davis, 1985; Vilenkin, 1985; Wilson and Silk, 1981; Wilson, 1983; Efstathiou and Silk, 1983; Bond and Efstathiou, 1984; Vittorio and Silk, 1984). Moreover, the independent evidence for dark matter is rather compelling (Rees, 1984).

The most plausible candidates for the dark matter can be treated as collisionless. If at a some epoch both the dark and the ordinary matter are nonrelativistic, it is reasonable to describe the mixture by a Vlasov-Euler-Poisson system. Quadratic nonlinear effects in such a model were considered recently by one of us (Kates, 1987). Disturbances whose characteristic length exceeds the Jeans length by a sufficient amount were found to satisfy to good accuracy the well-known Kadomtsev-Petviashvili equation of type I (Kadomtsev and Petviashvili, 1970), which can be solved exactly by inverse scattering and admits two-dimensional lump solutions. These lumps, which represent regions of *negative* density contrast, suggest themselves as possible mechanisms for cosmic voids. The model also predicts strong biasing, i.e., the density contrast of the ordinary matter in lumps is much larger than that of the dark matter.

Under what circumstances might a model consisting of two noninteracting fluids be relevant for cosmology? First, suppose at some epoch the universe can be thought of as containing primarily dark matter and one kind of visible matter, for example photons. If the velocity dispersion of the dark matter is sufficiently small, then it is reasonable to describe it as a pressureless fluid, i.e., dust. If the ordinary matter is described as an ideal fluid with pressure, ther the mixture is an important special case of our model.

Second, suppose that one ignores the presence of dark matter. Then the universe may be thought of as containing two noninteracting fluids: the baryons and the photons. Because of the different Z-dependences (Z = redshift) of the background densities of photons and baryons, the relative proportion evolves with time. As we shall see below, the relative proportion turns out to be one of the critical parameters for determining which modes are modulationally stable, which are modulationally unstable, and which are modulationally unstable with self-focusing.

We begin by constructing a homogeneous, isotropic background (Newtonian) cosmological model characterized by an expansion parameter $a(t)$, a total background density $\rho_0(t)$, and a cosmological constant Λ, satisfying

$$\frac{d^2a(t)}{dt^2}=\left[-\frac{4}{3}\pi G\rho_0(t)+\Lambda/3\right]a(t) \tag{1}$$

$$\frac{d}{dt}[\rho_0(t)\ a(t)^3]=0 \tag{2}$$

(The notation and terminology follow that of Peebles (1980) unless otherwise stated.) We suppose that the density fluctuations are small compared to the background density and that the peculiar velocities (that is, the velocities with respect to the background expansion) are small compared to the speed of light. If the material is strictly nonrelativistic, one obtains Newtonian fluctuation equations on a background satisfying an expansion law appropriate for nonrelativistic material. One may also incorporate the case in which one of the fluids is a photon fluid. Of course, in that case the expansion law (2) for the background density is modified.

We will concentrate our attention on two-dimensional flows and dependence on two spatial dimensions. The extension to three dimensions should be evident.

Imagining that the material consists of ordinary and dark matter (of course, the model is equally applicable to any two decoupled fluid components, visible or not), we denote the exact density of the ordinary matter by $n(x, y, t)\cdot\rho_0(t)$ and the exact density of dark matter by $N(x, y, t)\cdot\rho_0(t)$. (That is, n and N are the relative concentrations.) The background concentrations are denoted by $n_0(t)$ and $N_0(t)$, respectively. We suppose that the pressure gradients are strictly proportional to the density gradients.

As in the one-fluid case (Kates, 1986), it is convenient to measure the time in units of $T_0\equiv(4\pi G\rho_0(t_0))^{-1/2}$ and the spatial coordinates in units of a_0, the expansion factor at some time t_0, (such as recombination). Following Peebles (1980), we introduce a modified Newtonian potential, and we remove the dimensions with a_0 and T_0. Similarly, we express the peculiar velocities in units of a_0/T_0 and denote these scaled peculiar velocity vectors by $[u(x, y, t), v(x, y, t), 0]$ and $[U(x, y, t), V(x, y, t), 0]$ for the ordinary and dark matter, respectively.

In this paper, we proceed as if $a(t)$ and $\rho_0(t)$ were constants. These conditions can be achieved self-consistently by choosing a nonvanishing cosmological constant Λ so that Eqs. (1)–(2) admit solutions in which $a(t)$ and $\rho_0(t)$ are independent of the time (Einstein static universe). In the Conclusions, we will discuss the degree to which one may hope that important features of the dynamics presented here persist when one allows for a realistic expansion law.

The system to be studied consists of mass-conservation and Euler equations for each fluid constituent, together with a Poisson equation for the modified potential, whose source is the sum of the density contrasts of the ordinary and dark matter. (We assume that the rotation of each velocity field venishes.) We thus seek functions $n(x, y, t)$, $N(x, y, t)$, $u(x, y, t)$, $v(x, y, t)$, $U(x, y, t)$, $V(x, y, t)$, and $\phi(x, y, t)$ satisfying

$$\partial_t(n)+\partial_x(nu)+\partial_y(nv)=0 \tag{3}$$

$$\partial_t(u)+u\partial_x u+v\partial_y u+c^2\partial_x(\log n)+\phi_x=0 \tag{4}$$

$$\partial_t(v)+u\partial_x v+v\partial_y v+c^2\partial_y(\log n)+\phi_y=0 \tag{5}$$

$$\partial_t(N)+\partial_x(NU)+\partial_y(NV)=0 \tag{6}$$

$$\partial_t(U)+U\partial_x U+V\partial_y U+C^2\partial_x(\log N)+\phi_x=0 \tag{7}$$

$$\partial_t(V)+U\partial_x V+V\partial_y V+C^2\partial_y(\log N)+\phi_y=0 \tag{8}$$

$$\partial_x^2\phi+\partial_y^2\phi=(n-n_0)+(N-N_0) \tag{9}$$

The quantities c^2 and C^2 are constants representing typical propagation velocities (squared) in these units and are proportional to the temperatures of the fluids (idealized as ideal gasses with an isothermal equation of state; the final results would differ only slightly for other equations of state). The two fluids are coupled only by gravitation. Note that the background corresponds to the solution $N=N_0$, $n=n_0$, with all peculiar velocity components and the potential vanishing. In a realistic background, of course, the time would enter Eqs. (3)–(9) explicitly.

2. Linear theory

Before we study the effects of nonlinearities, it is useful to obtain the dispersion relation describing the simple linearized theory of disturbances about the background:

$$(S+N_0)(R+n_0)=n_0N_0 \tag{10}$$

$$R\equiv\omega^2-c^2k^2 \tag{11}$$

$$S\equiv\omega^2-C^2k^2 \tag{12}$$

A special case is sketched in Fig. 1.

One branch (the "upper" branch) of the hyperbola defined by (10) always passes through the origin, the other ("lower" branch) passes through the point $(-2n_0, 2N_0)$. The allowed portions of each branch are of course those lying in the region of the R–S plane corresponding to $k^2>0$. Restricting the discussion to these allowed portions, we note that the upper branch always corresponds to modes with real frequency. The lower branch contains (in general) a stable and an unstable (imaginary-frequency) part. [If one of the fluid components is pressureless (dust), *all* of the lower branch represents imaginary frequencies.] The linear instability will be referred to below as the Jeans instability.

Along the asymptotes $S=-N_0$, $R=-n_0$ in the allowed regions, the dispersion relation approaches that of a single fluid with given density and sound speed.

If both frequencies corresponding to a given wavenumber are real, there are in general two propagating modes with distinct group velocities. We will be especially interested in those instabilities which are not present in the linear theory but do occur when nonlinearities are taken into account. Moreover, as we will see below, it is possible that the growth of linearly (Jeans-) unstable modes may saturate or slow down due to nonlinear effects. Thus, even at wavenumbers where the linear theory predicts *Jeans* instability, the dominant instability may be the nonlinear one.

Of particular interest is the high-mass-contrast case: Suppose that one of the fluids has a very low concentration compared to the other, i.e., $n_0/N_0 \ll 1$ (see Fig. 1). If $c^2>C^2$, the asymptotes

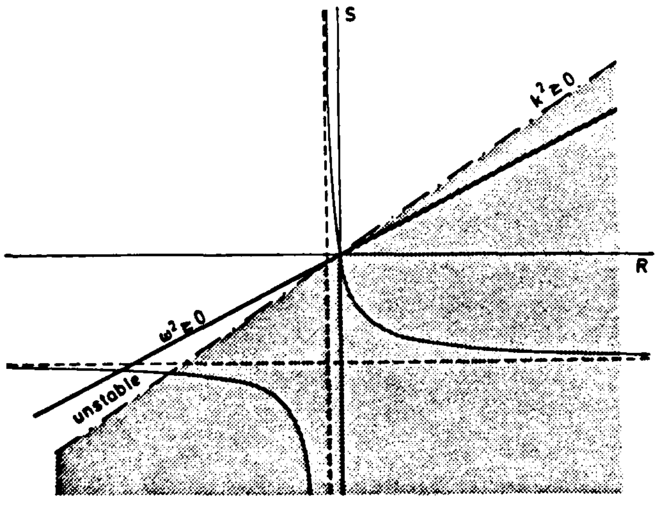

Fig. 1. Sketch of dispersion relation (2.1)–(2.3). (3.21) for the case $n_0=0.1$, $N_0=0.9$, $C^2/c^2=0.7$. The unshaded area designates the allowed region where $k^2>0$. The upper branch is automatically Jeans-stable ($\omega^2>0$). The lower branch contains an unstable domain $\omega^2<0$. Asymptotes are at $R=-n_0$, $S=-N_0$. The region shown corresponds roughly to $-2<R<2$, $-2<S<2$

cross in the unphysical region corresponding to $k^2<0$. Each branch remains near one asymptote, and one of the asymptotes is close to the origin. The lower branch satisfies approximately

$$\omega^2=-1+C^2k^2 \tag{13}$$

and will be referred to below (occasionally) as the "principal" section, because it resembles the one-fluid dispersion relation. The upper branch satisfies approximately

$$\omega^2=c^2k^2 \tag{14}$$

and will be referred to below as the "acoustic" section, because it passes through the origin like the dispersion relation for acoustic waves in the absence of gravitation. If $c^2<C^2$, the asymptotes cross in the physical region at $k^2=C^2-c^2$. The two branches cannot intersect, of course, but rather interchange roles. In this case, the term "principal section" will mean either branch when it is near the $S=-N_0$ asymptote, while the term "acoustic section" will mean either branch when it is near the $R=-n_0$ asymptote. Finally, in the high-mass-contrast case, the critical Jeans wavenumber k_J approaches $1/C$ as n_0/N_0 approaches zero.

Thus, the linear theory predicts that the solution of Eqs. (3)–(9) for generic initial conditions will be dominated at large times by the unstable modes of longest wavelength, which in the high-mass-contrast case have a growth rate of about unity (in units of the characteristic time scale T_0). The basic length scale which is singled out by the dynamics (as opposed to the initial conditions) is the Jeans length.

Since the Jeans length is presumably a decreasing function of time in our universe, the prediction one would obtain by extrapolating the *linearized* version of our model would seem to be that, at a given mass scale, growth can only occur after a well-defined epoch. As we saw above, the largest mass scales would become unstable first. However, there is no *justification* for expecting the linearized theory to provide an accurate approximation to the full theory over arbitrarilily long times. (Moreover, the limitations introduced by neglecting the expansion and by assuming a nonrelativistic model become more serious at long times and large lengths). In what follows, we will investigate the possible role of nonlinearities. As we shall see, at a given mass on length scale, it is possible for nonlinear instability to switch on "before" linear instability.

3. Derivation of the two-dimensional nonlinear Schroedinger equation for acoustic disturbances

In a weakly nonlinear system, nonlinear effects can become comparable with linear ones only if they accumulate over long times or distances. The strength of the nonlinearity thus introduces new length and time scales into the problem. In order to discuss weak, nearly plane-symmetric, nearly monochromatic fluctuations, we assume that the physical quantities of interest can be developed in asymptotic expansions depending on the coordinates and the small amplitude parameter ε in a prescribed way. Writing the ε-dependence as

$$n\sim n_0+\varepsilon n_1+\varepsilon^2 n_2+\varepsilon^3 n_3+\ \ldots \tag{15}$$

$$u\sim \varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3+\ \ldots \tag{16}$$

$$v\sim \varepsilon^2 v_2+\varepsilon^3 v_3+\ \ldots \tag{17}$$

$$N\sim N_0+\varepsilon N_1+\varepsilon^2 N_2+\varepsilon^3 N_3+\ \ldots \tag{18}$$

$$U \sim \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \ldots \tag{19}$$

$$V \sim \varepsilon^2 V_2 + \varepsilon^3 V_3 + \ldots \tag{20}$$

$$\phi \sim \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \ldots, \tag{21}$$

we note that the leading deviation of all quantities from their background values are of $O(\varepsilon)$ or higher.

As has been discussed in various papers and text books (Van Dyke, 1964; Cole, 1968) on two-timing methods, one demands, not only that the expansions (15)–(21) should represent pointwise valid asymptotic expansions, but also that the validity be uniform over some possibly ε-dependent domain of validity which may become large as $\varepsilon \to 0$. In this case, the domains of validity should extend over distances of at least order $1/\varepsilon$ (times a wavelength) and over time intervals of at least order $1/\varepsilon^2$ (times a period) at some fixed position with respect to the "wave packet". We introduce the slowly varying spatial coordinates

$$X \equiv \varepsilon x \tag{22}$$

$$Y \equiv \varepsilon y, \tag{23}$$

as well as the slow and slower time variables

$$T \equiv \varepsilon t \tag{24}$$

$$\tau \equiv \varepsilon^2 t. \tag{25}$$

Derivatives of functions of the slow variables are evidently down by factors of ε or ε^2. Since we are interested in the evolution of slowly modulated wave trains, the dependence of all quantities of interest on the fast variables (t, x) is taken into account by rapidly varying phase functions, whereas the dependence on slow variables occurs in a multiplicative amplitude or envelope function. (It is this envelope which will turn out to satisfy the NLS equation in the slow variables.) Note that the y-dependence occurs only through the slow variable Y. In first order, the assumed form is thus

$$n_1 = m_1 e^{i\theta} + m_1^* e^{-i\theta} \tag{26}$$

$$u_1 = \mu_1 e^{i\theta} + \mu_1^* e^{-i\theta} \tag{27}$$

$$N_1 = M_1 e^{i\theta} + M_1^* e^{-i\theta} \tag{28}$$

$$U_1 = \nu_1 e^{i\theta} + \nu_1^* e^{-i\theta} \tag{29}$$

$$\phi_1 = f_1 e^{i\theta} + f_1^* e^{-i\theta}, \tag{30}$$

where m_1, M_1, μ_2, ν_1, f_1 etc. are functions of (X, Y, T, τ). It is convenient to introduce the notation

$$k \equiv \frac{\partial\theta}{\partial x} \qquad \omega \equiv \frac{\partial\theta}{\partial t} \tag{31}$$

for the derivatives of the rapidly varying phase. Note that ω is the negative of the physical frequency. [In principle, the frequency could be large compared to unity, as long as it is restricted to satisfy $\omega^2 \ll 1/\varepsilon$, in order that the higher-order terms be truly decreasing in order of magnitude (Kates, 1986). However, no generality is lost by treating the wavenumber and frequency as if they were just of order unity, as will be done here.]

Our procedure is to collect terms order by order, paying special attention to resonances: Now, at each order ε^p, the system assumes the form

$$LQ_p = S_p, \tag{32}$$

where L is a fixed linear operator acting on the $O(\varepsilon^p)$ parts Q_p of the expansions (15)–(21) and S_p is a nonlinear function of the terms Q_1, Q_2, Q_{p-1}, and their derivatives. Resonances occur whenever S_p contains terms which are not orthogonal to the homogeneous solutions of the self-adjoint operator L. Resonances would lead to secular growth and thus nonuniformity of the asymptotic expansions (15)–(21). In order to obtain *uniformly valid* expansions, we employ a multiple-scales technique, which consists in choosing the functional dependence of the envelopes in such a way as to render the sources S_p orthogonal to the homogeneous solutions of L. In this problem, this condition amounts to selecting those terms which are proportional to a linearized propagating wave solution and demanding that the sum of all such terms vanish. Note however that the *nonresonant* parts of S_p need not vanish, but rather will drive Q_p.

It is convenient for the moment to carry out the calculation with the Y-dependence suppressed. As expected, at $O(\varepsilon)$, the solution takes the form of the linear theory in one dimension:

$$m_1 = \frac{k^2 n_0 f_1}{\omega^2 - c^2k^2} \tag{33}$$

$$\mu_1 = -\frac{\omega}{k}\frac{m_1}{n_0} \tag{34}$$

$$M_1 = \frac{k^2 N_0 f_1}{\omega^2 - C^2k^2} \tag{35}$$

$$\nu_1 = -\frac{\omega}{k}\frac{M_1}{N_0} \tag{36}$$

$$1 + \frac{n_0}{\omega^2 - c^2k^2} + \frac{N_0}{\omega^2 - C^2k^2} = 0, \tag{37}$$

where the dependence of f_1 on the slow variables is as yet undetermined. (Note that (37) is equivalent to (10)–(12).)

At $O(\varepsilon^2)$, the solution contains contributions at the fundamental and twice the fundamental:

$$n_2 = (m_2^{(1)} e^{i\theta} + c.c.) + (m_2^{(2)} e^{2i\theta} + c.c.) \tag{38}$$

$$N_2 = (M_2^{(1)} e^{i\theta} + c.c.) + (M_2^{(2)} e^{2i\theta} + c.c.) \tag{39}$$

$$u_2 = (\mu_2^{(1)} e^{i\theta} + c.c.) + (\mu_2^{(2)} e^{2i\theta} + c.c.) \tag{40}$$

$$U_2 = (\nu_2^{(1)} e^{i\theta} + c.c.) + (\nu_2^{(2)} e^{2i\theta} + c.c.) \tag{41}$$

$$\phi_2 = (f_2^{(1)} e^{i\theta} + c.c.) + (f_2^{(2)} e^{2i\theta} + c.c.). \tag{42}$$

Without loss of generality, we set

$$f_2^{(1)} = 0. \tag{43}$$

The solutions for the terms proportional to $e^{2i\theta}$, $e^{-2i\theta}$ take the form

$$\mu_2^{(2)} = -\frac{\omega}{k}[m_2^{(2)}/n_0 - (m_1/n_0)^2] \tag{44}$$

$$\nu_2^{(2)} = -\frac{\omega}{k}[M_2^{(2)}/N_0 - (M_1/N_0)^2] \tag{45}$$

$$\frac{m_2^{(2)}}{n_0} = \frac{k^2 f_2^{(2)}}{\omega^2 - c^2k^2} + \frac{1}{2}\left[1 + \frac{2\omega^2}{\omega^2 - c^2k^2}\right]\left[\frac{m_1}{n_0}\right]^2 \tag{46}$$

$$\frac{M_2^{(2)}}{N_0} = \frac{k^2 f_2^{(2)}}{\omega^2 - C^2k^2} + \frac{1}{2}\left[1 + \frac{2\omega^2}{\omega^2 - C^2k^2}\right]\left[\frac{M_1}{N_0}\right]^2 \tag{47}$$

$$f_2^{(2)} = -\frac{n_0}{6k^2}\left[1+\frac{2\omega^2}{\omega^2-c^2k^2}\right]\left[\frac{m_1}{n_0}\right]^2 - \frac{N_0}{6k^2}\left[1+\frac{2\omega^2}{\omega^2-C^2k^2}\right]\left[\frac{M_1}{N_0}\right]^2. \tag{48}$$

The terms proportional to $e^{i\theta}$, $e^{-i\theta}$ yield several algebraic relations as well as the first secular condition:

$$\mu_2^{(1)} = -\frac{\omega}{k}\frac{m_2^{(1)}}{n_0} + \frac{i}{n_0k^2}[-\omega\partial_X + k\partial_T]m_1[X, Y, T, \tau] \tag{49}$$

$$m_2^{(1)} = \frac{2ik\omega}{(\omega^2-c^2k^2)^2}[-\omega\partial_X + k\partial_T]f_1[X, Y, T, \tau] \tag{50}$$

$$v_2^{(1)} = -\frac{\omega}{k}\frac{M_2^{(1)}}{N_0} + \frac{i}{N_0k^2}[-\omega\partial_X + k\partial_T]M_1[X, Y, T, \tau] \tag{51}$$

$$M_2^{(1)} = \frac{2ikw}{(\omega^2-C^2k^2)^2}[-\omega\partial_X + k\partial_T]f_1[X, Y, T, \tau] \tag{52}$$

$$\partial_X(f_1) + \frac{D_2}{\omega}[\omega\partial_X - k\partial_T]f_1 = 0, \tag{53}$$

where we define for convenience

$$D_L \equiv \frac{n_0\omega^{2L-2}}{(\omega^2-c^2k^2)^L} + \frac{N_0\omega^{2L-2}}{(\omega^2-C^2k^2)^L}. \tag{54}$$

The secular condition (53) is a property of the linear theory and implies that, to a first approximation, the amplitude information propagates along the integral curves of the vector field

$$\sigma = \frac{\partial}{\partial T} + v_g\frac{\partial}{\partial X} \tag{55}$$

$$v_g = -(1+D_2)\frac{\omega}{k}, \tag{56}$$

where $v_g \equiv d(-\omega)/dk$ is the group velocity of the wave packet.

At $O(\varepsilon^3)$, the equations split into parts proportional to $e^{\pm 3i\theta}$, $e^{\pm 2i\theta}$, and $e^{\pm i\theta}$. The parts of the $O(\varepsilon^3)$ solutions proportional to $e^{\pm 3i\theta}$ and $e^{\pm 2i\theta}$ exert no influence on the secular terms at this order and will play no role in what follows. Let us denote the parts proportional to $e^{\pm i\theta}$ by $m_3^{(1)}$, $M_3^{(1)}$, $\mu_3^{(1)}$, $v_3^{(1)}$, $f_3^{(1)}$, etc. as before. We collect only those terms in the equations proportional to $e^{\pm i\theta}$. Using the mass conservation, Euler, and Poisson Eqs. (3)–(9) and setting (without loss of generality) $f_3^{(1)} \equiv 0$, we obtain the relations

$$n_0\mu_3^{(1)} = -\frac{\omega}{k}m_3^{(1)} + \frac{i}{k}[\partial_T m_1 + \partial_T m_2^{(1)} + n_0\partial_X\mu_2^{(1)}] - m_1^*\mu_2^{(2)} - m_2^{(2)}\mu_1^* \tag{57}$$

$$\begin{aligned} m_3^{(1)} = {} & n_0\left[\frac{m_1^*}{n_0^2}m_2^{(2)} - \frac{m_1^2m_1^*}{n_0^3}\right] + \frac{i}{k}\partial_X m_2^{(1)} \\ & + \frac{n_0}{\omega^2-c^2k^2}\left[k^2\mu_1^*\mu_2^{(2)} - k\omega\left[\frac{m_1^*}{n_0}\mu_2^{(2)} + \frac{m_2^{(1)}}{n_0}\mu_1^*\right]\right] \\ & + \omega^2\left[\frac{m_1^2m_1^*}{n_0^3} - \frac{m_1^*m_2^{(2)}}{n_0^2}\right] - \frac{i\omega^2}{k}\partial_X\left[\frac{m_2^{(1)}}{n_0}\right] \\ & - ik[\partial_T\mu_2^{(1)} + \partial_T\mu_1] + i\omega\left[\partial_T\left[\frac{m_1}{n_0}\right] + \partial_T\left[\frac{m_2^{(1)}}{n_0}\right] + \partial_X\mu_2^{(1)}\right] \end{aligned} \tag{58}$$

$$N_0v_3^{(1)} = -\frac{\omega}{k}M_3^{(1)} + \frac{i}{k}[\partial_T M_1 + \partial_T M_2^{(1)} + N_0\partial_X v_2^{(1)}] - M_1^*v_2^{(2)} - M_2^{(2)}v_1^* \tag{59}$$

$$\begin{aligned} M_3^{(1)} = {} & N_0\left[\frac{M_1^*}{N_0^2}M_2^{(2)} - \frac{M_1^2M_1^*}{N_0^3}\right] + \frac{i}{k}\partial_X M_2^{(1)} \\ & + \frac{N_0}{\omega^2-C^2k^2}\left[k^2v_1^*v_2^{(2)} - k\omega\left[\frac{M_1^*}{N_0} - v_2^{(2)} + \frac{M_2^{(1)}}{N_0}v_1^*\right]\right] \\ & + \omega^2\left[\frac{M_1^2M_1^*}{N_0^2} - \frac{M_1^*M_2^{(2)}}{N_0^2}\right] - \frac{i\omega^2}{k}\partial_X\left[\frac{M_2^{(1)}}{N_0}\right] \\ & - ik[\partial_T v_2^{(1)} + \partial_\tau v_1] + i\omega\left[\partial_\tau\left[\frac{M_1}{N_0}\right] + \partial_\tau\left[\frac{M_2^{(1)}}{N_0}\right] + \partial_X v_2^{(1)}\right] \end{aligned} \tag{60}$$

$$-\partial_X^2 f_1 + m_3^{(1)} + M_3^{(1)} = 0. \tag{61}$$

Substituting (57) and (59) into (61) and using (33), (34), (52) and (54), we obtain the following expressions, without the term involving $\partial_Y\partial_Y(f_1)$ in braces:

$$\begin{aligned} & \frac{2ik^2}{\omega}D_2\partial_\tau f_1 + \frac{k^6}{\omega^4}\left[2D_5 + D_4 - \frac{1}{2}D_3 - \frac{2}{3}D_2D_3 - \frac{1}{6}[D_2]^2 - \frac{2}{3}[D_3]^2\right]f_1^*(f_1)^2 + \frac{1}{\omega^2}[D_2 - 4D_3] \\ & \times[-\omega\partial_X + k\partial_T]^2 f_1 + \frac{2D_2}{\omega}[\omega\partial_X - k\partial_T]\partial_X f_1 - \partial_X\partial_X f_1 + \\ & \{-[1+D_2]\partial_Y\partial_Y(f_1)\} = 0. \end{aligned} \tag{62}$$

Using the secular condition (53) we find (again without the term in braces)

$$\begin{aligned} & \frac{2ik^2}{\omega}D_2\frac{\partial f_1}{\partial\tau} \\ & + \frac{k^6}{\omega^4}\left[2D_5 + D_4 - \frac{1}{2}D_3 - \frac{2}{3}D_2D_3 - \frac{1}{6}[D_2]^2 - \frac{2}{3}[D_3]^2\right]f_1^*(f_1)^2 \\ & + \left[\frac{-3(D_2)^2 + D_2 - 4D_3}{[D_2]^2}\right]\frac{\partial^2 f_1}{\partial X^2} - \left\{[1+D_2]\frac{\partial^2 f_1}{\partial Y^2}\right\} = 0. \end{aligned} \tag{63}$$

Without the term in braces, Eq. (63) is the one-dimensional nonlinear Schroedinger equation. However, it is possible to generalize the above calculation in order to take into account the Y-dependence as follows: The *linear* terms in (63) are those which would be obtained by applying the present singular perturbation method to the linearized version of (3)–(9). It can be easily verified that the *nonlinear* term is unaffected by the inclusion of Y-derivatives and y-components of velocity. We have therefore carried out a second calculation of the linear coefficients, including Y-derivatives, by constructing the linear operator associated with the dispersion relation (10) and expanding in terms of slow derivatives using the computer algebra system MUMATH. This procedure confirms the coefficients obtained above and provides the Y-coefficients in braces in (62)–(63).

Finally, letting

$$F = \frac{k^2}{\omega^2}f_1 \tag{64}$$

$$\xi = kX \tag{65}$$

$$\eta = kY \tag{66}$$

$$\bar{\tau} = \omega\tau, \tag{67}$$

we obtain the two-dimensional nonlinear Schroedinger equation in the form

$$\frac{i\partial F}{\partial \bar{\tau}} + 2W[F^* F^2] + D\frac{\partial^2 F}{\partial \xi^2} + E\frac{\partial^2 F}{\partial \eta^2} = 0, \tag{68}$$

where

$$W \equiv \frac{1}{4D_2}\left[2D_5 + D_4 - \frac{1}{2}D_3 - \frac{2}{3}D_2D_3 - \frac{1}{6}[D_2]^2 - \frac{2}{3}[D_3]^2\right] \tag{69}$$

$$D \equiv \frac{1}{2[D_2]^3}[D_2 - 3[D_2]^2 - 4D_3] \tag{70}$$

$$E \equiv -\frac{1}{2}[1 + D_2]/D_2. \tag{71}$$

The result of our multiple time-scales approach is thus the following: Provided F and its complex conjugate F^* are chosen in such a way as to satisfy Eq. (68) and its conjugate, no secular terms will arise in the solutions up to and including $O(\varepsilon^3)$, as is necessary if our expansions (15)–(21) are to represent *uniformly valid* asymptotic approximations (over the distance and time scales considered) to some exact solutions of the original problem (3)–(9) corresponding to particular initial value data. [Eq. (68) is a *necessary* condition; whether it is *sufficient* for uniformity remains an open question.]

4. Qualitative properties of the two-dimensional NLS equation

The degree to which the nonlinear theory differs qualitatively from the linear one depends on the relative signs of the coefficients D, E, and W. Since any solution of the one-dimensional NLS equation also solves the two-dimensional version, it is convenient to review a few of the properties of the one-dimensional NLS equation expressed in the form

$$\frac{i\partial F}{\partial \bar{\tau}} + 2WF^*F^2 + D\frac{\partial^2 F}{\partial \xi^2} = 0. \tag{72}$$

If W and D take different signs, then solitons do not form and the underlying acoustic disturbance is called "modulationally stable" (i.e., small modulations of a plane wave are always stable), and the predictions of the nonlinear theory are not much different from those of the linearized theory. If N and D take the same sign, then the acoustic disturbance is said to be "modulationally unstable" (i.e., there exist unstable perturbation modes of a plane wave), and soliton formation is possible.

Now, the one-dimensional NLS should give a good description of the dynamics during some time period if either one of the terms of (72) involving spatial derivatives is small compared to the other. Thus, if the transverse dependence is slow, even with respect to Y, then solitons can be expected if $W \cdot D > 0$. On the other hand, it is also possible in principle for the disturbances to be so coherent that the X-variations are much smaller than the Y-ones, and in this case "transverse" solitons would be possible if $W < 0$. (Note that $E < 0$ always holds in this system.)

Although of course solutions of the one-dimensional NLS equation also satisfy the two-dimensional version, they are unstable over very long time scales with respect to slowly varying or small perturbations in the second spatial direction (Ablowitz and Segur, 1981). Therefore, solitons should not be expected to persist indefinitely, even if they do form. Unfortunately, the general solution of (68) is not yet known. However, solutions of (68) satisfy a sequence of integral identities, the first few of which are

$$\frac{\partial J_0}{\partial \bar{\tau}} = 0, \tag{73}$$

$$\frac{\partial J_1}{\partial \bar{\tau}} = 0, \tag{74}$$

$$\frac{\partial^2 J_2}{\partial \bar{\tau}^2} = 8J_1, \tag{75}$$

where

$$J_0 \equiv \iint F^* F \, d\xi d\eta \tag{76}$$

$$J_1 \equiv \iint [D \cdot [F_\xi]^2 + E \cdot [F_\eta]^2 - WF^2F^{*2}] d\xi d\eta \tag{77}$$

$$J_2 \equiv \iint [D\xi^2 + E\eta^2] F^* F \, d\xi d\eta. \tag{78}$$

If the coefficients D, E and W are all of the same sign (say positive) then, for certain nonsingular initial data, Eq. (68) exhibits self-focusing singularities in a finite time (Zakharov and Synakh, 1976), as one can make plausible by the following argument: Suppose the solution for F were regular at all times. For the case $D \cdot E > 0$, the integrand of J_2 is positive definite. On the other hand, if the initial data has the property that the integral J_1 is negative, then by virtue of (74) it will remain so. However, Eq. (75) implies that a time $\bar{\tau}_0$ exists such that $J_2 < 0$, leading to a contradiction. The nature of self-focusing singularities has been investigated theoretically (Talanov, 1965; Zakharov and Synakh, 1976; Newell, 1978), and their occurrence has been observed in nonlinear optics (Chiao, 1964; Kelley, 1965).

Of course, only the *onset* of self-focusing is a true prediction of the theory presented here: If the density contrasts become large compared to unity, then the problem enters the fully nonlinear regime and the expansions assumed here are no longer appropriate. The essential point, however, is that the predictions of the nonlinear theory differ qualitatively from those of the linear theory: Even at wavenumbers where the *linear* Jeans instability is not present, the *nonlinear* theory predicts the possibility of soliton production or the possible occurrence of occasional dramatic pointlike increases in the density.

5. Regimes of stability, modulational instability, and self-focusing

It is routine to work out the symptotic values of the coefficients W, D, and E along both branches of the dispersion relation in the general case:

Along the principal section (lower branch), asymptotically in the limit $\omega^2 \to \infty$.

$$D_L = [-1]^L \left[\frac{\omega^2}{n_0}\right]^{L-1} + O[(\omega^2/n_0)^{L-2}] \tag{79}$$

$$E = -1/2 + O[(\omega^2/n_0)]^{-2} \tag{80}$$

$$D = 1/2\left[\frac{\omega^2}{n_0}\right]^{-1} + O[(\omega^2/n_0)^{-2}] \tag{81}$$

$$W = -\frac{2}{3}\left[\frac{\omega^2}{n_0}\right]^3 + O[(\omega^2/n_0)^2]. \tag{82}$$

The asymptotic values in the limit $\omega^2 \to \infty$ along the acoustic section (upper branch), can be obtained from (79)–(82) by replacing n_0 with N_0. Thus, at sufficiently high frequencies along both branches, D is always positive, W is always negative, there is no self-focusing, and in fact acoustic disturbances are modulationally stable with respect to the longitudinal direction. Since E is negative, the only possibility for soliton formation predicted here is the case where the acoustic disturbance is very coherent. Then, as described above, the one-dimensional NLS equation in the Y and τ variables admits soliton solutions.

In the limit $\omega^2 \to 0$ along the upper branch (acoustic section), the values approach

$$D_L = d_L k^{-2} + O(1) \tag{83}$$

$$D = -\frac{3}{2d_2}k^2 + O(k^4) \tag{84}$$

$$E = -\frac{1}{2} + O(k^2) \tag{85}$$

$$W = -\frac{1}{24}[2d_3 + d_2]^2 d_2^{-1} k^{-2}, \tag{86}$$

where

$$d_L \equiv \frac{(n_0 C^2 - N_0 c^2)^{L-1}}{(C^2 - c^2)^L}[n_0 + N_0]\left[\frac{1}{n_0^L} - + \frac{(-1)^L}{N_0^L}\right]. \tag{87}$$

Thus, all three coefficients are negative and self-focusing can occur. Note that the coefficient W diverges like k^{-2}.

Finally, as $\omega^2 \to 0$ along the lower branch (principal section), we obtain

$$D_L = \omega^{2L-2}(-1)^L \bar{d}_L + O(\omega^{2L}) \tag{88}$$

$$D = \frac{1}{2(\bar{d}_2)^2 \omega^4} + O(\omega^{-2}) \tag{89}$$

$$E = -\frac{1}{2\omega^2(\bar{d}_2)} + O(1) \tag{90}$$

$$W = \frac{\omega^2}{24\bar{d}_2}[3\bar{d}_3 - (\bar{d}_2)^2] + O(\omega^4) > 0, \tag{91}$$

where

$$\bar{d}_L \equiv (-1)^L \frac{\dfrac{n_0}{c^{2L}} + \dfrac{N_0}{C^{2L}}}{\left[\dfrac{n_0}{c^2} + \dfrac{N_0}{C^2}\right]^L}. \tag{92}$$

Self-focusing does not occur here, but since $W \cdot D > 0$, the system is modulationally unstable with respect to the longitudinal direction, and solitons may form.

For reasons described in Sect. 1, it is also of interest to study the high-mass-contrast case ($n_0/N_0 \ll 1$) in detail. In that case, we can develop the coefficients in powers of n_0/N_0 along either section. The cross-over region (which occurs only if $c^2 < C^2$) requires special care.

Along the principal section (which by definition means outside the cross-over region), the frequency is approximately

$$\omega^2 = -1 + C^2 k^2 + n_0 \Delta + O[(n_0/N_0)^2], \tag{93}$$

where

$$\Delta \equiv \frac{1}{[C^2 - c^2]k^2 - 1}. \tag{94}$$

The coefficients are then given approximately by

$$D_L = (-1)^L \omega^{2L-2} + \ldots \tag{95}$$

$$W = [\omega^2(1 + 5\omega^2 - 8\omega^4)/12] + \ldots \tag{96}$$

$$D = c^2k^2/(4\omega^4) + \ldots \tag{97}$$

$$E = -c^2k^2/(2\omega^2) + \ldots. \tag{98}$$

Along the acoustic section, the frequency satisfies approximately

$$\omega^2 = c^2k^2 + n_0\delta + \ldots \tag{99}$$

$$\delta \equiv \frac{(C^2 - c^2)k^2}{1 - (C^2 - c^2)k^2}, \tag{100}$$

and the coefficients are approximately

$$D_L = (\omega^2/n_0)^{L-1}\delta^L + \ldots \tag{101}$$

$$N = [\omega^2/(n_0\delta)]^3[2/3 - 1/(6k^2(C^2 - c^2))] + \ldots \tag{102}$$

$$D = -[(n_0\delta^3/(2\omega^2)][1 + 3/(k^2(C^2 - c^2))] + \ldots \tag{103}$$

$$E = -\frac{1}{2} + \ldots. \tag{104}$$

The limiting values along both branches for large and small frequency agree with those found above in the general case.

In the cross-over region, both hyperbolae are near the asymptotes, and a different expansion is called for. To leading "order" in (n_0/N_0),

$$\omega^2 = c^2k^2 + (n_0)^{1/2}\Gamma + \ldots \tag{105}$$

$$(C^2 - c^2)k^2 = 1 + 2(n_0)^{1/2}d + \ldots \tag{106}$$

$$\Gamma = d \pm (1 + d^2)^{1/2}, \tag{107}$$

with the upper (lower) sign corresponding to the upper (lower) branch, (see Fig. 1). The coefficients become

$$D_2 = \omega^2(1/\Gamma^2 + 1) + \ldots \tag{108}$$

$$D_L = (\omega^2/n_0)^{L-1}(n_0)^{1/2}/\Gamma^L + \ldots (L > 3) \tag{109}$$

$$N = \omega^6/(2\Gamma^3(\Gamma^2 + 1)n_0^{3/2}) + \ldots \tag{110}$$

$$D = -2\Gamma^3/(\omega^2(n_0)^{1/2}(\Gamma^2 + 1)^3) + \ldots. \tag{111}$$

In the one-fluid limit, which is discussed in the Appendix, the three coefficients W, D, and E are never all of the same sign and self-focusing never occurs.

It is also of interest to examine the effects of nonlinearities on the linearly (Jeans-) unstable modes where $\omega^2 < 0$. Now the interpretation of a "secular" condition changes of course when the underlying disturbance has imaginary frequency. However, it is still necessary to enforce the secular conditions (55) and (63) in order to keep the higher-order terms in (15)–(21) from growing larger than the lower-order ones (e.g., the function $\varepsilon^3 t e^{\gamma t}$ would eventually grow larger than $\varepsilon e^{\gamma t}$). Taking $\omega \equiv i\gamma$, and noting that

$$\gamma^2 = 1 - c^2k^2, \tag{112}$$

we examine the implications of the conditions (55) and (63) as

$k \to 0$, which is where the linear instability grows the fastest. The conditions become

$$i\hat{c}_T f_1 = 0 \tag{113}$$

$$\gamma \hat{c}_T f_1 + 2k^4 e^{2\gamma t} f_1^* f_1^2 + \frac{1}{2} C^2 [\hat{c}_X^2 + \hat{c}_Y^2] f_1 = 0. \tag{114}$$

Equation (114) is a (nonlinear) diffusion equation, and gradients tend to get smoothed out with time. The nonlinear term would then have a tendency to cause f_1 to saturate. Ignoring spatial derivatives, we find that the quantities $f_1^* f_1$ and the potential ϕ approach

$$\lim_{t \to \infty} [f_1^* f_1] = \frac{1}{2} \frac{\gamma^2}{k^4 \varepsilon^2} e^{-2\gamma t} \tag{115}$$

$$\lim_{t \to \infty} [\phi] = \sqrt{2}\gamma/k^2 \cos(k(x - x_0)). \tag{116}$$

However, by this time the growth would have proceeded so far that the asymptotic expansions (15)–(21) and the approximation scheme used to derive (114) would no longer be valid. Nevertheless, the onset of saturation *is* predicted by these considerations, and therefore it is not valid to assume that the linear Jeans instability is the dominant one, even if its initial growth rate is larger than that of the nonlinear instability. The modes with $k \to 0$ appear to reach the largest amplitudes before saturating.

5. Conclusions

According to Eqs. (84)–(86) and (102)–(104), self-focusing occurs when $\omega^2 \to 0$ along the acoustic section. "Longitudinal" modulational instability can occur on the principal sections for small to moderate values of ω^2, whereas the crossover regions (for the case $(n_0/N_0) \ll 1$) are modulationally stable. "Transverse" modulational instability can occur at sufficiently high frequencies along both sections. For certain values of the parameters, it is also possible for self-focusing to occur along the acoustic section at a wavenumber for which both the acoustic and principal modes are stable. This means that in an expanding universe, where the Jeans length is in fact slowly decreasing, a wave of fixed k will be subject to modulational instability and self-focusing before the linear Jeans instability sets in.

It is of interest to estimate the number of solitons of self-focusing singularities that might be expected to occur for a given set of initial conditions. The evolution of a soliton from some initial data typically requires the initial pulse envelope to have some region over which the phase of the pulse is fairly constant and which contains a normalized area of at least $\pi/2$. Over such a region, the typical number of solitons N_s is the normalized area divided by π. Our estimate thus becomes

$$N_s \approx (W/D)^{1/2} \frac{k^2}{\omega^2} [\varepsilon \mu_0], \tag{117}$$

where μ_0 is roughly the size of the envelope measured in the original coordinates, which is related to the degree of coherence of the initial conditions and depends on some correlation length. Along both branches for large ω, N_s diverges like ω^4. Along the acoustic section for $\omega \to 0$, N_s diverges like ω^{-2}, along the principal section like ω^{-3}. For moderate values of ω, however, soliton formation is unlikely unless $\mu_0 \gtrsim O[\varepsilon^{-1}]$.

The typical time for the nonlinearity to have an effect depends on W and is roughly

$$(\Delta t)_{\mathrm{NL}} \sim \frac{\omega^3}{W k^4 \varepsilon^2}, \tag{118}$$

whereas the typical time for the dispersion to have an effect is roughly

$$(\Delta t)_{\mathrm{D}} \sim \frac{4\pi^2 \mu_0^2}{D\omega}. \tag{119}$$

Along either branch we find, for $\omega \to \infty$,

$$(\Delta t)_{\mathrm{NL}} \sim \varepsilon^{-2} \omega^{-7} \tag{120}$$

$$(\Delta t)_{\mathrm{D}} \sim \mu_0^2 \omega. \tag{121}$$

Along the acoustic section for $\omega^2 \to 0$

$$(\Delta t)_{\mathrm{NL}} \sim \omega \varepsilon^{-2} \tag{122}$$

$$(\Delta t)_{\mathrm{D}} \sim \mu_0^2 \omega^{-3}, \tag{123}$$

while along the principal section for small positive frequencies

$$(\Delta t)_{\mathrm{NL}} \sim \omega \varepsilon^{-2} \tag{124}$$

$$(\Delta t)_{\mathrm{D}} \sim \mu_0^2 \omega^3. \tag{125}$$

Note that it is possible to achieve short nonlinear growth time scales for both sufficiently small and sufficiently large frequencies. Since the number of solitons estimated above also grows at large and small frequencies, it seems likely that modulational instability and self-focusing will play a role in a more realistic theory which includes the background time-dependence. For moderate frequencies, growth times are long compared to the time scale for the expansion of the universe. However, even in this case it is possible that (as in the case of the linear Jeans instability) the qualitative predictions of our "frozen" background will shed some light on what is to be expected in a realistic background.

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Appendix: the one-fluid limit

In Kates (1986), the two-dimensional NLS equation was first derived for a one-fluid model. Unfortunately, the coefficients given there were incorrect. Eqs. (34)–(38) of that paper should read

$$C \equiv 2 - \frac{4}{3} c_s^2 k^2 \tag{34}$$

$$D \equiv \frac{\omega}{3k} [4c_s^2 k^2 - 3] \tag{35}$$

$$E \equiv -\frac{1}{2k^2} + \frac{1}{3} c_s^2 \tag{36}$$

$$-2i\omega a_\tau - \frac{c_s^2}{\omega^2} a_{\xi\xi} + c_s^2 a_{YY} + Q a^2 a^* = 0 \tag{37}$$

$$Q \equiv \frac{8}{3}\omega^4 - \frac{5}{3}\omega^2 - \frac{1}{3} = \frac{8}{3}c_s^4k^4 - 7c_s^2k^2 + 4 \tag{38}$$

(Eqs. (35) and (38) were correct as given in the original paper)

Evidently, the coefficients never take the same sign, and self-focusing therefore does not occur. However, modulational instability occurs for any ω. Q changes sign at around $k_{NL} = 1.336\,k_J$, as reported in the original paper. Below this critical wavenumber, "longitudinal" solitons are permitted. Above k_{NL}, "transverse" solitons are permitted.

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Lattice Equations and Integrable Mappings[1]

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1. Nonlinear integrable lattices (i.e. integrable partial *difference* equations,cf e.g. [1, 3]) are of fundamental importance for the study of classically integrable systems. They are generic in the sense that their various continuous limits give rise to the hierarchies of integrable PDE's [4]. Furthermore, their study opens up some new points of view on classical integrability in general [5]. In this note we report on another application of such systems. In fact, we will show how these lattices give rise to nonlinear integrable mappings. Such mappings are of interest for the investigation of various aspects of dynamical systems (e.g. to study bifurcations, transition to chaos, perturbation techniques, cf. e.g. [7]).

Probably the oldest nonlinear integrable mapping is the elliptic billiard due to C.G.J. Jacobi [6]. More recently E.M. McMillan found a four-parameter family of rational mappings of the plane, together with their invariants [8]. An eighteen-parameter family generalizing the one of McMillan was presented by G.R.W Quispel et al. in [9]. Moreover, a connection with soliton equations of differential-difference type was established, cf. also [10]. However, a spectral interpretation of the integrability of these mappings on the basis of a Lax pair was lacking.

In this note we take a different point of view from the one expounded in [9, 10] by considering integrable lattices rather than differential- difference equations as a starting point. This is convenient, because it allows us to obtain the mappings in a more natural way than before, namely not as special reductions, but from the consideration of an initial value problem on a two-dimensional planar lattice. Furthermore, we shall show that these

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mappings do indeed carry a spectral interpretation, and that the invariants can be systematically constructed from a Lax pair.

2. We shall use as a prime example for the exposition of our ideas the lattice KdV equation. This equation (as well as lattice analogs of other integrable PDE's) was obtained in [1] using a discrete version of the direct linearization method introduced in [2].

The equation reads

$$(p-q+\hat{u}-\tilde{u})(p+q-\hat{\tilde{u}}+u) \;=\; p^2-q^2 \,. \tag{1}$$

In (1) $u=u(n,m)$ is the dynamical variable at the lattice site (n,m), $n,m\in\mathbf{Z}$, the $\tilde{}$ and $\hat{}$ are shorthand notations for translation on the lattice, i.e. $\tilde{u}=u(n+1,m)$, $\hat{u}=u(n,m+1)$ and p and q are the lattice parameters $p,q\in\mathbf{C}$. Eq. (1) arises as the compatibility condition of a pair of linear problems (Lax pair) defining the shifts (translations) of an eigenfunction Ψ_k (k being the spectral parameter) in the n- and m-directions,

$$(p-k)\tilde{\Psi}_k \;=\; L_k\cdot\Psi_k \;,\; (q-k)\hat{\Psi}_k \;=\; M_k\cdot\Psi_k \;, \tag{2}$$

where L_k is given by

$$L_k \;=\; \begin{pmatrix} p-\tilde{u} & 1 \\ k^2-p^2+(p+u)(p-\tilde{u}) & p+u \end{pmatrix} , \tag{3}$$

and where M_k is given by a similar matrix obtained from (3) by making the replacements $p\to q$ and $\tilde{}\;\to\;\hat{}$.

Let us now consider an initial value problem for (1) on the lattice. One way of doing this is to assign initial data on a "staircase" as in Fig. 1. From the fact that eq. (1) involves only the four variables situated on the four lattice sites around a simple plaquette, it follows that the information on these staircases evolves diagonally through the lattice along "parallel" staircases. Hence, the initial value problem is well-defined.

Consider,now,the case of periodic initial values along the staircases. The simplest nontrivial example is drawn in Fig. 1. where we have period 2 initial data on two diagonals, a,b,c,d denoting the different initial values for u. By applying the lattice equation we can calculate the data on all the diagonals where the (multiple) primes denote the various iterations of this procedure. One way of doing this is by regarding the first iteration as a vertical shift on the lattice, thereby "updating" the values of a,b,c,d as follows: $b'=c$, $d'=a$, and a' and c' are calculated in terms of a,b,c,d using (1).

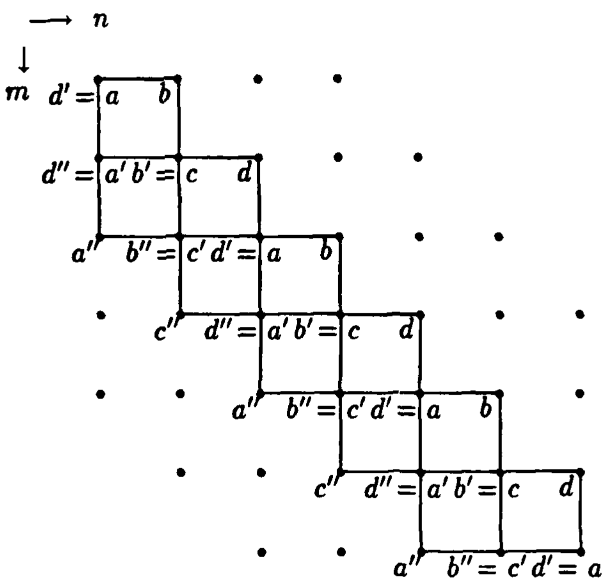

Fig 1. Periodic configuration of initial data on the lattice and its iteration.

However,a reduction to a 2-dimensional one is readily obtained by considering the differences along the diagonals:$X = a - c, Y = b - d$ .The mapping reads

$$Y' = -X \ , \ X' = Y + \frac{2\epsilon\delta X}{\epsilon^2 - X^2} \ , \tag{4}$$

where $\delta = p - q$ and $\epsilon = p + q$. Eq. (4) is the original McMillan map [8] which was recovered as a reduction of the discrete Nonlinear Schrödinger equation in [10]. This map arises as the compatibility condition of a Lax pair that can be constructed from (2) as follows .

We introduce an ordered product of the Lax matrices (3) along the staircase. Because of the periodicity it is sufficient to consider a product of only four of them, i.e.

$$\mathcal{L} = M_{ad} \cdot L_{dc} \cdot M_{cb} \cdot L_{ba} \tag{5}$$

where L_{ba} and L_{dc} denote the matrices L of (3) in which we substitute $u \to a, \tilde{u} \to b$ respectively $u \to c, \tilde{u} \to d$, and similarly in M we substitute $u \to b, \hat{u} \to c$ and $u \to d, \hat{u} \to a$ to obtain M_{cb} and M_{ad} respectively. The matrix $\mathcal{L}$ can be regarded as a *monodromy* matrix with invariant spectrum since

$$\mathcal{L}' = M_{a'a} \cdot \mathcal{L} \cdot M_{a'a}^{-1} \ . \tag{6}$$

In particular we have that $tr(\mathcal{L})$ is invariant under the mapping, and this indeed gives us the McMillan invariant [8]

$$\mathcal{I} = \left(\epsilon^2 - X^2\right)\left(\epsilon^2 - Y^2\right) + 2\epsilon\delta X Y \ , \tag{7}$$

which can be parametrized in terms of Jacobi elliptic functions.

3. We have shown how to obtain in a natural way an integrable mapping from the lattice KdV equation (1). Of course this is only a simple example, and there are various ways to generalize our procedure. First of all, we can obtain higher-dimensional mappings related to the lattice KdV by considering higher periods in the initial data. In the case of period 3 for example this yields a 4-dimensional mapping with *two* invariants that can be obtained using a monodromy matrix as above. Another generalization is to consider other types of staircases, generally some discrete curves on the lattice. Apart from the lattice KdV one might consider other existing lattice equations as starting point, such as the lattice MKdV, the discrete-time Toda equation, the lattice BSQ and MBSQ equations [11] and so on. This work is in progress [12]. It would be of interest to investigate the canonical structure of the mappings and of the lattices from which they are obtained. For related work towards this direction see e.g. [13].

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